Axiomatic choice models and duality

Weddepohl, Hubertus Nicolaus

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AXIOMATIC CHOICE MODELS

H. N. WEDDEPOHL

UNIVERSITAIRE PERS ROTTERDAM

STELLINGEN

I

Overgang naar een nieuw en beter stelsel van sociaal-economische orde is slechts mogelijk na (interdisciplinair) wetenschappelijk onderzoek op grote schaal en na wetenschappelijk begeleide experimenten. Het op gang komen van dergelijk onderzoek is van groot belang.

II

Als men ervan uitgaat, dat de overheid niet tot taak heeft het aantal wisselingen van werkkring door werknemers te minimaliseren, maar wel om deze wisselingen zo soepel mogelijk te doen verlopen, dan verdient het aanbeveling om ook inkrimpingen en bedrijfssluitingen van marginale industriële bedrijven van grotere omvang in perioden van gunstige conjunctuur te bevorderen.

III

De voornaamste taak van de economische wetenschap met betrekking tot de economische politiek, is het presenteren van een verzameling keuzemogelijkheden in ook voor leken begrijpelijke taal. Economen bezitten evenwel geen bijzondere bevoegdheid tot het aanbrengen van een voorkeursordening tussen die mogelijkheden.

IV

De handel in homogene goederen, zoals die met name op beurzen plaatsvindt, zou aanzienlijk kunnen worden vereenvoudigd indien de prijsvorming aan computers zou worden toevertrouwd.

V

In de huidige samenleving is de directe invloed die economische beslissingen uitoefenen op het inkomen van degenen die de beslissingen nemen, minder, naarmate de bedragen die met de beslissingen gemoeid zijn, toenemen.

VI

De nauwe band van de wiskunde met de natuurwetenschappen vormt een belemmering voor een wederzijdse bevruchting van wiskunde en maatschappijwetenschappen.

VII

Zo weinig zinvol als het zou zijn de faculteit der wiskunde en natuurwetenschappen op te splitsen in afzonderlijke faculteiten voor natuurkunde, scheikunde, biologie enz., even weinig zinvol is de huidige verdeling van de maatschappijwetenschappen over verschillende faculteiten.

VIII

Het is gewenst dat het wetenschappelijk corps van elke instelling van wetenschappelijk onderwijs zodanig wordt opgebouwd, dat een aanzienlijk deel van zijn leden is afgestudeerd aan een andere dan de eigen instelling.

IX

Voor de verkiezing van vertegenwoordigende lichamen van universiteiten, verdient het personenstelsel de voorkeur boven een lijstenstelsel.

X

Het is gebruikelijk politieke stromingen af te beelden op een lijn die van links naar rechts loopt. Dit model zou winnen aan realiteitswaarde, hoewel wellicht niet aan duidelijkheid, indien men de uiteinden van de lijn zou verbinden, zodat een cirkel ontstaat.

Stellingen behorende bij het proefschrift van H. N. Weddepohl: Axiomatic Choice Models.

Axiomatic choice models

Axiomatic choice models
and duality

Proefschrift

ter verkrijging van de graad van doctor in de economische wetenschappen aan de Katholieke Hogeschool te Tilburg, op gezag van de rector magnificus dr. C. F. Scheffer, hoogleraar in de bedrijfshuishoudkunde, in het openbaar te verdedigen in de aula van de hogeschool op 20 mei 1970 des namiddags om 4 uur

door

Hubertus Nicolaus Weddepohl

geboren te Hilversum

1970
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PROMOTOR: PROF. DR. J.J.J. DALMULDER

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Contents

List of symbols

Logic

$\wedge$	and	$a \wedge b$	both statements a and b are true
$\vee$	(inclusive) or	$a \vee b$	at least one of a and b is true
$\Rightarrow$	implication	$a \Rightarrow b$	if a is true, then b is true
$\Leftrightarrow$	logical equivalence	$a \Leftrightarrow b$	if and only if a is true, then b is true
$\forall$	universal quantifier	$\forall x \in X$: $a(x)$	for every $x \in X$, the statement $a(x)$ is true
$\exists$	existential quantifier	$\exists x \in X$: $a(x)$	there exists $x \in X$, such that $a(x)$ is true
$\nexists$		$\nexists x \in X$: $a(x)$	there does not exist $x \in X$, such that $a(x)$

Set theory

$\in$	element	$x \in A$	x is an element of the set A
$\notin$		$x \notin A$	x is not an element of the set A
		$A \cup B$	the set of points that are in at least one of A and B
$\cup$	union	$\bigcup_{i \in I} A_i$	$= A_1 \cup A_2 \ldots$
		$A \cap B$	the set of points that are both in A and in B
$\cap$	intersection	$\bigcap_{i \in I} A_i$	$= A_1 \cap A_2 \ldots$
$\subset$	inclusion	$A \subset B$	A is contained in B
$\not\subset$		$A \not\subset B$	A is not contained in B
$\supset$		$A \supset B$	$\Leftrightarrow B \subset A$
$-$	difference	$A - B$	the set of points that are in A, but not in B
$\varnothing$	the empty set		
$\{\ \}$		$\{x, y\}$	the set consisting of the points x and y
		$\{x \mid a(x)\}$	the set of points x, such that the statement $a(x)$ is true.

Other symbols

$d(x, y)$	distance between two points x and y	def. 4.2.1.
$\|x-y\|$	$= d(x, y)$	
$\|x\|$	$= \|x-0\|$, length of vector x	
$B_\epsilon(x)$	spherical neighbourhood of x with radius ϵ	def. 4.2.4.
$\text{Int}\, A$	interior of the set A	def. 4.2.7.
$C\,\|\,A$	closure of the set A	def. 4.2.13.
$\text{Bnd}\, A$	boundary of the set A	def. 4.2.11.
$\text{Conv}\, A$	convex hull of the set A	def. 4.2.24.
$P \times X$	cartesian product of P and X	section 4.2.
$\prod_{i \in I} X_i$	$= X_1 \times X_2 \times \cdots$	
$\alpha \geqslant \beta$	α is at least as great as β ($\alpha \leqslant \beta \Leftrightarrow \beta \geqslant \alpha$)	
$\alpha \ngeqslant \beta$	α is not at least as great as β	
$\alpha > \beta$	$\Leftrightarrow \alpha \geqslant \beta \wedge \beta \ngeqslant \alpha$	
$x \geqq y$	$\forall i$: $x^i \geqslant y^i$ ($x \leqq y \Leftrightarrow y \geqq x$)	(2.2.16)
$x \geq y$	$\Leftrightarrow x \geqq y \wedge y \ngeqq x$	(2.2.17)
$x > y$	$\forall i$: $x^i > y^i$	(2.2.18)
R	set of real numbers	
R^n	n-dimensional Euclidean space	section 4.2.
$R_+{}^n$	the non negative part of R^n	(4.3.1)
N	set of natural numbers	
$[0, 1]$	$= \{\lambda \in R \mid 0 \leqslant \lambda \leqslant 1\}$, unit interval	
$f: X \to Y$	the mapping f of X into Y	def. 2.3.1.
$F: X \to Y$	the correspondence F of X into Y	def. 2.3.2.

Elements of sets and single valued functions are denoted by small letters $(x, a, u(x))$.
Sets and correspondences are denoted by capital letters $(X, A, F(x))$.
Sets of sets are denoted by script letters $(\mathscr{P})$.
Greek letters denote real numbers (α, λ).

Preference and favourability relations (Ch. III–VI)

$\gtrsim$ ($\succ$, $\sim$)	preference	def. 3.2.1.
R (P, I)	direct revealed preference	def. 3.4.1.
R^k (P^k, I^k)	revealed preference in k steps	def. 3.4.10.
$\bar{\mathrm{R}}$ ($\bar{\mathrm{P}}$, $\bar{\mathrm{I}}$)	indirect revealed preference	def. 3.4.11.
$\bar{\bar{\mathrm{R}}}$ ($\bar{\bar{\mathrm{P}}}$, $\bar{\bar{\mathrm{I}}}$)	extended revealed preference	def. 5.9.17.
$\gtrsim^*$, ($\succ^*$, $\sim^*$)	favourability	def. 3.5.1.
R^*, (P^*, I^*)	direct revealed favourability	def. 3.5.4.
R^{*k}, (P^{*k}, I^{*k})	revealed favourability in k steps	def. 3.5.11.
$\bar{\mathrm{R}}^*$, ($\bar{\mathrm{P}}^*$, $\bar{\mathrm{I}}^*$)	indirect revealed favourability	def. 3.5.12.
$\bar{\bar{\mathrm{R}}}^*$, $\bar{\bar{\mathrm{P}}}^*$, $\bar{\bar{\mathrm{I}}}^*$	extended revealed favourability	def. 5.9.17.

1. Introduction

A THE SUBJECT OF THE STUDY

1.1 THE CONCEPT OF CHOICE

If it is said that somebody chooses something, it is meant that an individual intends to do something in order to reach certain goals that have value for him, while he could also do something else, which would have other consequences, that are valued differently.
With choosing are connected three notions: individuals, activities and results.
Individuals: It is always somebody who chooses and who does this more or less consciously. In fact this 'individual' can be a single person, but also a group, e.g. a board of directors, a family or a parliament.
Activities are actions that have to be performed in order to realise the choice. These can be very simple, like indicating a piece of cake on a dish, or very complicated, like the realisation of an investment project.
Results are *all* valuable consequences of an activity, like the disposal of a piece of cake or the profits of an investment. Among the results may figure favourable and unfavourable ones.
Henceforward, all actions connected with a choice are considered to be a single activity and all consequences that follow an activity are considered to be a single result.
If an individual chooses, he will perform a certain activity, in order to get a result. Now activities can be connected with results in different ways:

a. The individual's activities are the only ones that have a 'conscious' influence on the results. This can still occur in two ways:
aa. an activity is followed with certainty by a definite result
ab. an activity can be followed by different results, each having a definite probability (known or unknown), hence the result is partially determined by chance.

Examples

Somebody has a guilder and he can spend it among other things on:
aa. an ice cream
activity: buying of the ice cream
result: disposal of the ice cream, loss of the guilder
ab. a lottery ticket
activity: buying of the ticket and eventually cashing of the prize
result: loss of the guilder(if the ticket is a blank)
loss of the guilder and disposal of the prize (if the ticket is winning)

b. There exist also choice problems in which different persons are implicated in such a way that each can independently determine what activity he will realise, but where the result depends also on the activities performed by the others. Thus the result depends on what combination of activities occurs. If a participant tries to get the best result for himself, he must attempt to predict what activities the others will perform, before choosing his own activity. (Clearly now also chance may influence the results) Problems of this type are analysed in the theory of games.[1]

Example

A simple game may illustrate this (borrowed from Luce and Raiffa, page 95). This game is known as the prisoner's dilemma. Two criminals are guilty of a crime, but the evidence is not sufficient to convict them. It is however proved that they also have committed a minor crime. It is known that the judge will take their behaviours into account: if both do not confess they will get the maximum punishments for the minor crime, if one confesses he will get a small punishment, if both confess their punishment will be lesser than the maximum. The two criminals cannot com-

1. See e.g. VON NEUMANN and MORGENSTERN, LUCE and RAIFFA.

municate with each other. Each has two activities, confess and not confess. The results are given in the following table:

		criminal B	
		not confess	confess
crim. A	not confess	A: 1 year B: 1 year	A: 10 years B: 6 months
	confess	A: 6 months B: 10 years	A: 3 years B: 3 years

In this study we shall not deal with game theory. With every activity is connected a result or a probability distribution of results. A choice is made on the basis of results and the way to obtain them is the related activity.

For a person who has to choose, the choice problem has two different aspects. In the first place he must find out what are the results he can obtain. In the second place he needs a procedure to select from the available results a best one or, what is the same, to select a best activity from the feasible ones.

By the *set of feasible activities* of an individual is meant the collection of all activities the individual can perform, whereas one and only one of the activities of the set must be performed. With every element of the set of feasible activities corresponds a result or a probability distribution of results. The *choice set* is the collection of all results that are consequences of an activity of the set of feasible activities. An element of this set is either a single result or a distribution of results. One and only one of the results can be obtained by the individual. If he chooses a result of the set, he automatically determines an activity. Therefore we shall concentrate on choice sets.

Remarks

1. A choice set can be compared with the set of elementary events in a statistical experiment. There an impersonal stochastical mechanism selects one and only one event, whereas in a choice problem a person consciously selects a result or an activity.
2. In game theory it is more efficient to define choice sets for the

collection of players. An element of this set now consists of a result for every player, corresponding to a certain combination of activities of the players.

3. If the individual can decide to do nothing, 'doing nothing' must be an element of the set of feasible activities.

If an individual has to choose, he will ask if the choice set contains an element that can be considered as a best one and if such a result exists, it will be chosen. In this case choice is equivalent to the determination of a best result. What 'best' means, depends on the problem at hand. It can mean 'nicest' or 'most profitable' or 'most efficient' etc. A best element must be better than other elements. The existence of the notion 'best' implies the existence of the notion 'better than'. In the next chapter conditions will be given that ensure the possibility to determine a best element. In order to anticipate difficulties that might arise if two results are indifferent, the analyses will be based on the notion 'better or equally good'. Clearly, if two results are considered to be equally good, it is irrelevant which one is chosen.

1.2 EXAMPLES OF CHOICE PROBLEMS IN ECONOMICS

Directly or indirectly, choice problems play an important part in nearly every economic theory. Directly if the choice problem as such is discussed, indirectly if the information, necessary for choice, is collected, either by the search for criterions to determine the best result, or by constructing a set of feasible activities of a choice set. The following examples may illustrate this:

Consumer

Consumer choice

Choice set: all combinations of quantities of commodities that can be bought with a certain income at given prices.
Best element: the combination of quantities of commodities that the consumer considers to be most attractive.

Producer

1 *Production problem*

Set of feasible activities: all production techniques that are available to produce a certain quantity of some commodity.
Best element: the activity that entails minimal cost.

2 *Investment problem*

Set of feasible activities: all appropriate investment projects.
Best element: If receipts and expenditures are fixed: the project with highest present value, or the project with highest rate of return, or the project with the shortest pay back period. (Clearly, every criterion can lead to a different choice.) If the receipts and expenditures follow a probability distribution: the project with highest expectation of the present value.

Government

Budget problem

Set of feasible activities: all possible combinations of taxes and loans on one side and collective expenditures on the other side.
Best element: the activity that is considered most favourable by the authority that decides, e.g. a parliament.

In the above examples it is reasonable to suppose that only the activities of the individuals concerned affect the results. However in connection with investment problems game theoretical situations can appear, e.g. if in a small country a few big enterprises intend to invest in the same direction.
An obvious game theoretical problem is the following:

Duopoly

Set of feasible activities: all possible price fixations by each of the duopolists.
Choice set: the combinations of profits for each duopolist corresponding to the activities.

The question what is the best element of a choice set can only be answered if the goals of the individual are known. In the above examples some implicit assumptions about these goals have been made. In the examples concerning the producer, it was assumed that he wants to minimize cost or to maximize profits. The government can aim at maximal welfare (whatever that may be) for its subjects.

B THE METHOD OF THE STUDY

1.3 NORMATIVE AND DESCRIPTIVE THEORIES

In economics it is often desirable to make a distinction between normative and descriptive theories[2] and this is especially true in the theory of choice:
a. In a normative theory it is tried to solve the choice problem for the individual. It provides him with rules that allow him to detect what is the best of all feasible choices. *Insofar as* the theories of this book might be *interpreted* normatively, they do not give solutions, but only present conditions which 'best' must satisfy.
b. A descriptive theory however does not give rules, but only tries to formulate existing rules, in order to explain and eventually predict actual behaviour. Now the question is not 'how should the individual behave?' But 'how does he behave?'
The models presented in this book will be interpreted as descriptive theories. Descriptive economic theories try to explain economic processes and the result of any economic process is finally determined by decisions of individuals. Therefore in any economic theory at least implicit assumptions are made about choice behaviour. Obviously, which individuals decide (choose) depends on the economic system in which a process takes place. In a purely capitalistic economy decisions are made by families and entrepreneurs, in a purely socialist economy by government authorities.
Of course, also in normative theories choice plays an important part. Theories of economic policy and business economics try to formulate rules for government behaviour and business behaviour.
It should be noted that from a formal point of view no difference between the two types of theories exists. Every theory can be interpreted normatively as well as descriptively. The way in which the theory is founded is decisive. Note that a normative theory becomes descriptive if individuals behave in accordance with the rules.

2. See KOOPMANS (1957), p. 134.

1.4 THE AXIOMATIC METHOD

An *economic theory* consists of a collection of logically coherent statements on economic phenomena. If such a theory is formulated in mathematical terms, it is called an *economic model.*[3]

Besides the rather vague concept of a theory that is used by economists, logicians use a slightly different, but well defined concept of a deductive theory.

'We may characterise a *deductive science* (or *deductive theory*) T as being the set of all statements – usually called the theorems of T – which can be derived, starting from a certain set of fundamental statements – usually called the *axioms*, *postulates*, or *hypotheses* underlying the deductive science T – by means of logical inference. Likewise T is characterised by certain specific notions; these notions are either *primitive*, or else they must be *defined* in terms of the primitive notions.'[4]

The axioms on which the theory is based may not contradict each other and it is desirable that they are independent, i.e. that one axiom cannot be deduced from the other ones.[5]

Thus a deductive theory is purely formal. The primitive concepts have no connection with reality, neither have the axioms, which are expressed in terms of these concepts. However a deductive theory can be interpreted, i.e. a meaning can be given to the primitive notions.

If the primitive concepts are given an economic interpretation,[6] such that the deductive theory is a formalisation of an economic theory, the deductive theory becomes an economic model; since it is based on a set of axioms, it is called an axiomatic model.[7]

In this book we shall construct a number of axiomatic models on choice behaviour. We choose primitive notions that are interpreted as economic concepts and present axioms which are statements about properties of the primitive concepts. Then other properties of the primitive concepts are deduced.

The advantage of an axiomatic treatment is that all assumptions of the

3. See SUPPES (1961), p. 169.
4. BETH, p. 81.
5. BETH, p. 82.
6. On the use of axiomatics in economics, see WOLD (1953), p. 75 and KOOPMANS, p. 132.
7. We believe that the notion of a model used here corresponds to the one in use among economists. Logicians however use the term in a somewhat different sense. See SUPPES (1961).

model are fixed exactly, clearly and completely[8] by the axioms. A second advantage is that the same deductive theory can be given different interpretations, so that it constitutes an economic model on different subjects.

From what has been said it will be clear that the meaning of axioms in economics and in any other science on reality is different from the one in mathematics. In mathematics axioms treat a constructed 'reality', a 'reality' that is precisely constructed by the axioms, and the question if they have something to do with reality can be left open, since it is not necessary to interpret them. The application of the theory in another science requires interpretation.

Between axiomatic models of a normative and a descriptive theory only a difference exists with respect to the way in which the axioms are founded:

a. If *all* axioms are meant to describe a reality, the model is descriptive. If the reality actually behaves according to the axioms, the model is a true description and hence the theorems are other true statements on reality.

b. If *at least one* of the axioms is a *rule* according to which reality ought to behave, the model is normative. (Other axioms might be descriptive.) The theorems are other rules now.

Obviously, from a descriptive theory alone it is impossible to deduce normative conclusions, neither can normative conclusions be applied as if they are descriptive.

The application of axiomatics is very old; it goes back as far as Euclid. However only during the last century or so axiomatic theories of other branches of mathematics and logic besides plane geometry, have been constructed. In economics and other social sciences the application of axiomatics is much more recent.[9]

Remark

Instead of the word 'axiom' also 'postulate', 'assumption' or 'hypothesis' are used. Note that an 'axiom' is *definitely not* an 'absolutely true statement'!

8. However, it is also assumed that results from other disciplines are valid, particularly the rules of logical inference and the applied mathematical procedures. (See TARSKI, p. 119).

9. See e.g. WOLD (1943), p. 89 and VON NEUMANN and MORGENSTERN, p. 617.

1.5 DEDUCTION AND INDUCTION

Pure deduction is impossible in economics. Economic theory treats a certain aspect of reality and therefore it is impossible to proceed by deduction alone. Only if one creates his own reality one can abandon induction, and that is only possible in mathematics. Precisely because one intends to do economics, judgements about reality are included and it is claimed that these describe some reality. This claim can only be based on induction, which in social sciences often takes the form of intuition.

In the construction of axiomatic models, the phases of induction and deduction are strictly separated. The choice of axioms is based on intuitive reasoning and observation, both being inductive processes. The derivation of theorems from the given set of axioms is a deductive process. Note that a theorem can eventually establish that the set of axioms is inconsistent or that one axiom can be deduced from other axioms.

Since axioms cannot be proved, a (descriptive) axiomatic model always bears a certain degree of uncertainty. (Only in the few cases a reality is completely known, axioms simply represent observed facts: induction by complete addition.[10]) Obviously, it is desirable to examine if a theory is verified in reality. This can be done by comparing both axioms and derived theorems with observed facts and here we have another inductive process.

10. VAN LAER, p. 56.

2. Mathematical concepts for choice theory

2.1 INTRODUCTION

In 1.1 we introduced the concept of a choice set and we announced to examine under what conditions a best element can be defined for this set. However before proceeding to this, we first introduce another notion.

The choice space is the set of all elements that might possibly be elements of a choice set, hence the choice space has all possible choice sets as subsets. If for example the choice set of a consumer consists of all combinations of quantities of commodities that can be bought with a certain income at given prices, then the choice space contains all possible combinations of quantities of commodities and hence any choice set is a subset of this space. Clearly the choice space is also defined if the choice set is not (exactly) known. The choice space will be denoted by the symbol X, where $x \in X$ and $y \in X$ indicate that x and y are elements of the choice space (instead of the word 'element' we also use the word 'point').

In the next section ordering relations will be treated and we shall show how these can be used in connection with the notion 'better than' etc. In 2.3 we shall define mappings and correspondences and in 2.4 'best elements' will be discussed. Finally we shall introduce order preserving functions and utility functions.

2.2 ORDERING RELATIONS

If for every couple of elements $\{x, y\}$, where x and y are elements of a set X, a certain statement about these two elements, in the given order, can only be true or false, this statement establishes a *binary relation* on X. If this relation is denoted by the symbol R, we have $x\mathsf{R}y$ if the statement is true and $x\not\mathsf{R}y$ if the statement is false.[1]

Examples 2.2.1

If X is the set of all human beings, then the statement 'be father of' establishes a binary relation on the set X. Now $x\mathsf{R}y$ means 'x is father of y' and $x\not\mathsf{R}y$ means 'x is not father of y'. If X is a set of numbers, then 'greater than' is a binary relation, usually written $>$.

There exist many binary relations, but in this book we are only interested in a certain type of binary relations, namely *ordering relations*. Therefore we define a number of important properties.

Definition 2.2.2

If $x,y,z \in X$ and R is a binary relation on X, then R is said to be

a *transitive* if $x\mathsf{R}y \wedge y\mathsf{R}z \Rightarrow x\mathsf{R}z$
b *reflexive if* $\forall x \in X$: $x\mathsf{R}x$
c *symmetric* if $x\mathsf{R}y \Rightarrow y\mathsf{R}x$
d *a-symmetric* if $x\mathsf{R}y \Rightarrow y\not\mathsf{R}x$
e *anti-symmetric* if $x\mathsf{R}y \wedge y\mathsf{R}x \Rightarrow x = y$
f *complete* if $\forall x,y \in X$: $x\mathsf{R}y \vee y\mathsf{R}x$

A binary relation is said to be an ordering relation if it is transitive, that is: if the relation is true for x and y and also for y and z, then it *must* also be true for x and z (in the given order). If $x\mathsf{R}y$ is read 'x is greater than y', transitivity means:

$$\text{if} \begin{cases} x \text{ is greater than } y \\ y \text{ is greater than } z \end{cases} \quad \text{then also } x \text{ is greater than } z$$

Clearly this is always true. But it is less evident that the following always holds:

$$\text{if} \begin{cases} x \text{ is better than } y \\ y \text{ is better than } z \end{cases} \quad \text{then also } x \text{ is better than } z$$

In fact at the end of this section we shall show that there are cases where one is inclined to say 'better than', while the transitivity condi-

1. See BERGE (1959), p. 30, DEBREU (1959), chapter I, CHIPMAN (1960).

tion is not fulfilled.[2] Nevertheless it seems reasonable to suppose that in most cases where a binary relation means 'better than' or something like that, it is transitive. It would e.g. seem very odd if somebody would maintain 'I like apples more than pears, I like pears more than plums and I like plums more than apples'.[3]
Meanwhile numerous binary relations exist that are not transitive, e.g. the relation 'be cousin of'.
We shall define different types of ordering relations. All of them are transitive and satisfy also one or more of the other conditions introduced in definition 2.2.2. These relations can be read 'at least as good as', or 'better than or equivalent to' or 'preferred to',[4] however other interpretations are possible as will be shown by the examples. Mostly, we shall use the symbol $\gtrsim$ instead of R. From the relations $\gtrsim$ can be derived other binary relations, denoted by $\sim$ and $\succ$, where

$$\begin{aligned} &x \sim y \Leftrightarrow x \gtrsim y \wedge y \gtrsim x \\ &x \succ y \Leftrightarrow x \gtrsim y \wedge y \not\gtrsim x \end{aligned} \tag{2.2.3}$$

If $x \gtrsim y$ means 'x is at least as good as y', $x \sim y$ can be read 'x is equivalent to y', 'x is indifferent to y' or 'x is equally good as y', while $x \succ y$ means 'x is better than y' or 'x is *strictly* preferred to y'.[4] The relation $\sim$ holds between all elements of X, between which $\gtrsim$ is symmetric, and $\succ$ holds between the points of X, between which $\gtrsim$ is a-symmetric.

Definition 2.2.4

A binary relation on a set X is said to be a *partial preordering*, if it is *transitive* and *reflexive*.
Hence, it is required that (see definition 2.2.2):

$$x \gtrsim y \wedge y \gtrsim z \Rightarrow x \gtrsim z \qquad \text{(transitivity)}$$
$$\forall x \in X: x \gtrsim x \qquad \text{(reflexivity)}$$

In addition to transitivity, reflexivity is required. This condition means that the binary relation always holds between any point x and itself. Clearly this is always true if $\gtrsim$ means 'at least as good as.' Note that $\gtrsim$ now cannot be read 'better than'. That reflexivity holds or not is always

2. MAY (1954) gives other counterexamples.
3. See also SAVAGE (1954), p. 20.
4. Throughout this book 'preferred to' will include equivalence. If equivalence is excluded, we shall say 'strictly preferred'.

obvious. Only because of a possible lack of transitivity, it can be doubted if in a concrete choice problem a binary relation is a partial preordering. It should be observed, that a partial preordering does not require completeness, hence it is allowed that two elements are not comparable. Two points x and y can exist, such that $x \not\gtrsim y$ and $y \not\gtrsim x$, i.e. x is not at least as good as y, nor is y at least as good as x.

Example 2.2.5

Let X be the set of all investment projects for a producer and let $x \gtrsim y$ (x is preferred to y) be true if simultaneously:
1. the rate of return of x is not lower than that of y,
2. the expenditure for x is not higher than that for y.
Now if of two projects one has higher rate of return and the other requires lower expenditure, the two are not comparable.

Example 2.2.6

The structure of a partial preordering can be elucidated by the following figure:

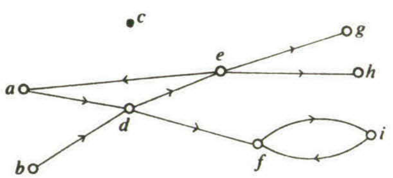


fig. 2.2.7

Let X be the set of points a through i. The arrows represent roads that can only be passed in the indicated directions. If $\gtrsim$ means 'can be reached from' $\gtrsim$ is a partial preordering.
1. $\gtrsim$ is transitive (e.g. $d \gtrsim a$, $a \gtrsim e$ and hence $d \gtrsim e$).
2. $\gtrsim$ is reflexive since every point can be reached from itself. Note that the isolated point c cannot be reached from any other point, nor can any other point be reached from c, hence c is not 'comparable' to any other point. The points i and h are not 'comparable' either.

A very important special case of a partial preordering is the equivalence relation.

Definition 2.2.8

A binary relation on a set X is said to be an *equivalence relation* if it is *transitive*, *reflexive* and *symmetric*.
This requires that we have (see definition 2.2.2)

$$x \sim y \wedge y \sim z \Rightarrow x \sim z$$
$$\forall x \in X: x \sim x$$
$$x \sim y \Rightarrow y \sim x$$

Hence an equivalence relation is a partial preordering, that satisfies the additional condition that it is symmetric, i.e. if the relation holds between x and y (in that order) it also holds between y and x. This relation can be read 'equivalent to' etc.
From a partial preordering an equivalence relation can be derived in a natural way (see (2.2.3))

$$x \sim y \Leftrightarrow (x \gtrsim y \wedge y \gtrsim x)$$

or in words, x is equivalent to y, if we have simultaneously x is at least as good as y and y is at least as good as x.

Example 2.2.9

In example 2.2.6 we have $d \sim e$, $d \sim a$, $e \sim a$ and $f \sim i$. $e \sim d$ can be read 'e and d can be reached from each other'.

Example 2.2.10

Numerous equivalence relations are in use, e.g. equally large, equally old, live in the same town, brother or sister of, and also the identity $=$.
Note that we must have *exactly* equally large, etc. If also points that are approximately equally large were considered equivalent, it would be possible to construct a chain of elements, such that the first and the last were completely different.

From a partial preordering we derive a strict ordering relation, denoted by $\succ$, where (see (2.2.3))

$$x \succ y \Leftrightarrow x \gtrsim y \wedge y \not\gtrsim x$$

which, as already mentioned, can be read 'better than' etc. The above formula says:

x is better than y	if and only if	x is at least as good as y and *not* y is at least as good as x

Now $\succ$ has the following properties

$$\begin{array}{ll} x \succ y \Rightarrow y \not\succ x & \text{(a-symmetry)} \\ x \succ y \wedge y \succ z \Rightarrow x \succ z & \text{(transitivity)} \end{array} \qquad (2.2.11)$$

Clearly $\succ$ is not reflexive. If $x \gtrsim y$, we have either $x \sim y$, or $x \succ y$.

Example 2.2.12

In example 2.2.5 we have: project x is better than project y if either the rate of return of x is *higher* than that of y and the expenditure is *not higher*, or the rate of return is *not lower* and the expenditure is *lower*.

Definition 2.2.13

A binary relation on a set X is said to be a *partial ordering* if it is *transitive, reflexive* and *anti-symmetric*.
Hence it is required that (see definition 2.2.2)

$$\begin{array}{ll} x \gtrsim y \wedge y \gtrsim z \Rightarrow x \gtrsim z & \text{(transitivity)} \\ \forall x \in X: x \gtrsim x & \text{(reflexivity)} \\ x \gtrsim y \wedge y \gtrsim x \Rightarrow x = y & \text{(anti-symmetry)} \end{array}$$

A partial ordering is a partial preordering with anti-symmetry as an additional condition: if x is preferred to y and y is preferred to x, x and y must be identical, hence different equivalent elements do not exist.

Example 2.2.14

Figure 2.2.7 becomes a partial ordering if we drop the arrows that pass from i to f and from e to a. Thus we get figure 2.2.15. It is now impossible to follow a road that leads back to the point of departure.

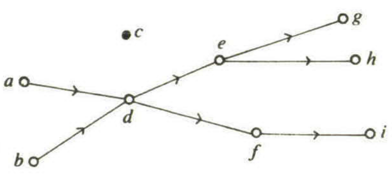


fig. 2.2.15

The euclidean space R^n of real vectors $x = (x^1, x^2, \ldots, x^n)$, (where x^i $(i = 1, 2, \ldots, n)$ is a real number) is partially ordered by the relation $\geqq$ (at least as great as) if we define

$$x \geqq y \Leftrightarrow \forall i: x^i \geqslant y^i \tag{2.2.16}$$

The derived relation (see (2.2.3)) is denoted by $\geq$, where

$$x \geq y \Leftrightarrow x \geqq y \wedge y \not\geqq x \Leftrightarrow \forall i: x^i \geqslant y^i \wedge \exists i: x^i > y^i \tag{2.2.17}$$

The symbol $>$ is used for cases that *all* components of x are greater than the corresponding components of y

$$x > y \Leftrightarrow \forall i: x^i > y^i \tag{2.2.18}$$

Definition 2.2.19

A binary relation on a set X is said to be a *complete preordering*, if it is *transitive* and *complete*.
This requires

$x \gtrsim y \wedge y \gtrsim z \Rightarrow x \gtrsim z$ (transitivity)
$\forall x, y \in X: x \gtrsim y \vee y \gtrsim x$ (completeness)

Note that reflexivity is implied by completeness, since x and y may be identical. Hence a complete preordering is a partial preordering with the completeness condition added. This condition requires that every couple of points of X are comparable: if $x, y \in X$ we have in any case x is preferred to y, or y is preferred to x.
From the complete preordering $\gtrsim$, the relations $\sim$ and $\succ$ are derived (see (2.2.3)). The first is an equivalence relation and the strict ordering relation $\succ$ is transitive and a-symmetric. Further we now have

$$\forall x, y \in X: x \succ y \vee x \sim y \vee y \succ x \tag{2.2.20}$$

Examples 2.2.21

a. Let X be the set of feasible investment projects of a producer. If $x \gtrsim y$ means that the present value of y is not higher than that of x, $\gtrsim$ on X is a complete preordering.
b. Let X be the set of all human beings and let $\gtrsim$ mean 'at least as old as'. Now X is completely preordered by $\gtrsim$.

Definition 2.2.22

A binary relation on a set X is said to be a *complete ordering*, if it is *transitive*, *anti-symmetric* and *complete*.
It is required that we have

$$x \gtrsim y \wedge y \gtrsim z \Rightarrow x \gtrsim z \qquad \text{(transitivity)}$$
$$\forall x,y \in X; x \gtrsim y \vee y \gtrsim x \qquad \text{(completeness)}$$
$$x \gtrsim y \wedge y \gtrsim x \Rightarrow x = y \qquad \text{(anti-symmetry)}$$

Hence, a complete ordering is a partial ordering which is also complete or a complete preordering which is also anti-symmetric.
The set R of real numbers is completely ordered by $\geqslant$ ('at least as great as') and from this relation $>$ ('greater than') is derived

$$x > y \Leftrightarrow x \geqslant y \wedge y \ngeqslant x$$

The binary relations, introduced above, are summarised in the following table:

2.2.23	transit	reflex.	symmetry	anti-symm.	compl.
equivalence relation	x	x	x		
part. preordering	x	x			
part. ordering	x	x		x	
complete preord.	x	*			x
complete ordering	x	*		x	x

*is implied by the other conditions

The connection between the four ordering relations can also be illustrated by the following figure, borrowed from Debreu.[5] The partial preordering is transitive and reflexive; the arrows indicate the conditions that are added.

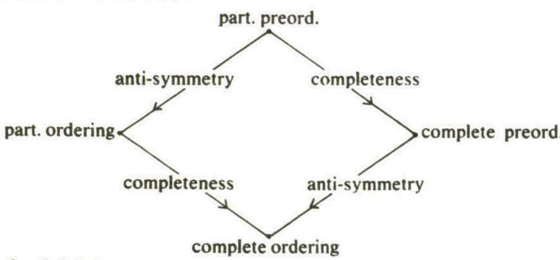


fig. 2.2.24

5. See Debreu (1959), p. 8.

Definition 2.2.25

If $\sim$ is an equivalence relation on X and $x \in X$, then the subset

$$E_x = \{y \in X | y \sim x\}$$

is an *equivalence class* of X.

A set can contain many equivalence classes. Equivalence classes associated with equivalent elements are identical

$$x \sim y \Rightarrow E_x = E_y \tag{2.2.26}$$

Since from every preordering an equivalence relation can be derived, every preordered set can be divided into equivalence classes and these are ordered if we define

$$E_x \gtrsim E_y \Leftrightarrow x \gtrsim y \tag{2.2.27}$$

The equivalence classes of an ordering all consist of one element (because of anti-symmetry).

Examples 2.2.28

a. The set X of example 2.2.5 can be divided into six equivalence classes: $\{a, e, d\}$, $\{b\}$, $\{c\}$, $\{f, i\}$, $\{g\}$, $\{h\}$. Points of the same class can be reached from each other. Further we have e.g. $\{h\} \gtrsim \{a, e, d\}$ and this means: the point h can be reached from any of the points a, e and d. The equivalence classes are now partially ordered.

b. In example 2.2.21 an equivalence class consists of projects having the same present value. These classes are completely ordered.

Remark 2.2.29

Cases can exist in which one is inclined to give the interpretation 'better than' to a binary relation, whereas it is not transitive. We have x better than y and y better than z, but it is not true that x is better than z. If the relation is denoted by R, we have

$x\mathsf{R}y$, $y\mathsf{R}z$ and $x\not\mathsf{R}z$ (but e.g. $z\mathsf{R}x$)

This may especially happen if a choice from a choice set must be made by a group of persons. If each member of the group has ordered

the set by a transitive binary relation, it seems obvious to consider x preferred to y if a majority prefers x to y. This however does not always lead to a transitive relation.[6]
An example may illustrate this:

> A family, consisting of father, mother and son John, has a T.V. set and it has been agreed that the program selected by a majority, will be watched. One night there are simultaneously three interesting programs: f (football match), q (quiz) and c (cowboy movie). The members of the family rank the programs in the following way
> father: $f \succ q \succ c$
> mother: $q \succ c \succ f$
> John: $c \succ f \succ q$
> Voting gives the following results:
> f against q :for f: father and John hence $f\mathrm{R}q$
> q against c :for q: father and mother hence $q\mathrm{R}c$
> Transitivity would now require:
> $f\mathrm{R}q \wedge q\mathrm{R}c \Rightarrow f\mathrm{R}c$
> but
> f against c :for c: mother and John hence $c\mathrm{R}f$

On this problem, also known as 'Condorcet's paradox', exists a vast literature[7] not so much in connection with T.V. problems as with respect to parliaments and boards of directors.
The lack of transitivity implies that no order preserving function on a choice set can be defined (see 2.4 below) and therefore, in general, a utility function for a group of consumers cannot be constructed.

2.3 MAPPINGS AND CORRESPONDENCES

Definition 2.3.1

A *mapping* f of a set X into a set Y (f: $X \to Y$) is a law that connects with every $x \in X$ *one and only one element* $f(x) \in Y$.
It is possible that $X = Y$, then f is a mapping from X into itself. A mapping is also called a function. If $Y = R$, f is said to be a *real valued function* and then to every $x \in X$ corresponds a real number.

6. See also RIJKEN VAN OLST, p. 105.
7. See ARROW (1951) and also MAY (1954).

Definition 2.3.2

A *correspondence* F of a set X into a set Y ($F: X \to Y$) is a law that connects with every element $x \in X$ a subset $F(x) \subset Y$.
Instead of correspondence, the terms multivalued mapping and set valued function are also used. In the following chapters, some correspondences will be called functions (choice function, demand function, price function).
If $C \subset X$ and $f: X \to Y$ and $F: X \to Y$, we have

$$\begin{aligned} f(C) &= \{y \in Y \mid \exists x \in X: y = f(x)\} \\ F(C) &= \{y \in Y \mid \exists x \in X: y \in F(x)\} \end{aligned} \tag{2.3.3}$$

If $f(X) = Y$ $(F(X) = Y)$, a mapping (correspondence) is said to be *onto* Y. Now every point of Y is connected with some point of X.
Correspondences have similar properties as mappings, e.g. continuity, monotonicity, etc. Besides this, for correspondences, also the properties of the sets $F(x)$ can be of interest, whereas, the point $f(x)$ has no other properties.

2.4 MAXIMAL ELEMENTS AND GREATEST ELEMENTS

If a binary relation is defined on a set X, it is also defined on any subset of X. If X is a choice space, (see 2.1) and P is a choice set, certain points of P can eventually be considered as 'best elements'. In this section it will be shown that two types of 'best elements' can exist.

Definition 2.4.1

If R is a binary relation on a set X and if R is transitive and $P \subset X$, then $x_0 \in P$ is said to be a *maximal* element of P, if

$$\forall y \in P: x_0 \mathrm{R} y \vee y \not\mathrm{R} x_0$$

(If $\forall y \in P: y\mathrm{R}x_0 \vee x_0 \not\mathrm{R} y$, x_0 is a minimal element)
The relation R may be any of the relations defined in 2.2. If $\succsim\, = \mathrm{R}$ is a preordering (partial or complete) and if $\succsim$ is read 'at least as good as', we may have

$x_0 \succsim y \wedge y \succsim x_0 \Leftrightarrow x_0 \sim y$	(x_0 is equivalent to y)
$x_0 \succsim y \wedge y \not\succsim x_0 \Leftrightarrow x_0 \succ y$	(x_0 is better than y)
$x_0 \not\succsim y \wedge y \not\succsim x_0$	(x_0 and y are not comparable)

But it is excluded that holds

$$x_0 \not\gtrsim y \wedge y \gtrsim x_0 \Leftrightarrow y \succ x_0 \qquad (y \text{ is better than } x_0)$$

Definition 2.4.2

If R is a binary relation on a set X and if R is transitive and $P \subset X$, then $x_0 \in P$ is said to be a *greatest* element of P, if

$$\forall y \in P: x_0 \mathrm{R} y$$

(If $\forall y \in P$: $y\mathrm{R}x_0$, x_0 is a smallest element)
If $\gtrsim = \mathrm{R}$ is a preordering, then we may only have

$$x_0 \gtrsim y \wedge y \gtrsim x_0 \Leftrightarrow x_0 \sim y \qquad (x_0 \text{ is equivalent to } y)$$
$$x_0 \gtrsim y \wedge y \not\gtrsim x_0 \Leftrightarrow x_0 \succ y \qquad (x_0 \text{ is better than } y)$$

But it is excluded that

$$x_0 \not\gtrsim y \wedge y \not\gtrsim x_0 \qquad (x_0 \text{ and } y \text{ are not comparable})$$
$$x_0 \not\gtrsim y \wedge y \gtrsim x_0 \Leftrightarrow y \succ x_0 \qquad (y \text{ is better than } x_0)$$

Hence, x_0 is a maximal element, if a better one does not exist in P and it is a greatest element, if it is at least as good as any other point of P.

Theorem 2.4.3

a. If R is a transitive binary relation on a set X, then every greatest element is a maximal element.
b. If R is a complete preordering, every maximal element is a greatest element.

Proof

(a) $x_0\mathrm{R}y \Rightarrow x_0\mathrm{R}y \vee y\not\mathrm{R}x_0$.
(b) If R is complete, we have $y\not\mathrm{R}x_0 \Rightarrow x_0\mathrm{R}y$,
hence $x_0\mathrm{R}y \vee y\not\mathrm{R}x_0 \Leftrightarrow x_0\mathrm{R}y$.

A partially preordered set can have maximal elements which are not greatest elements.

Example 2.4.4

In figure 2.2.7, the points c, g, h, i, and f are maximal elements, where $f \sim i$, while the others are not comparable. The points a, b

and c are minimal elements. In figure 2.2.15, c, g, h, and i are maximal elements.

A completely ordered set can only have one maximal element. If a set is completely *pre*ordered, it can have different *equivalent* maximal elements.

Definition 2.4.5

A set $\mathscr{A}$ is called the *power set* of a set X, if

$$P \in \mathscr{A} \Leftrightarrow P \subset A$$

Hence, the power set is the set of all subsets of X.
If X is preordered, with every element of $\mathscr{A}$, i.e. with every subset of X, can be associated its maximal elements. We define a correspondence $H : \mathscr{A} \to X$.

Definition 2.4.6

If X is a set, preordered by a relation $\gtrsim$ and $\mathscr{A}$ is its power set, we have for every $P \in \mathscr{A}$:

$$H(P) = \{x \in P | y \in P \Rightarrow x \gtrsim y \vee y \not\gtrsim x\}$$

Assume $\gtrsim$ is a complete preordering. Then by theorem 2.4.3, $H(P)$ is the set of greatest elements of $P \in \mathscr{A}$. It is possible that $H(P) = \varnothing$ for some $P \in \mathscr{A}$, i.e. P does not contain a greatest element. Two different cases can be distinguished:
1. $H(P) = \varnothing$ and no point $x \in X$ exists such that $x \gtrsim y$ for every $y \in P$ and hence, the set X does not contain a point that is at least as good as the elements of P. It is said that the set P has no *upper bound* with respect to the relation $\gtrsim$ and P is *not bounded*. In this case the set X is not bounded either: $H(X) = \varnothing$.

Example 2.4.6

Let $X = \{x \in R | x \geqslant 0\}$: X is the set of non negative real numbers. Let $\gtrsim$ be the relation $\geqslant$. Now $H(X) = \varnothing$ and for $P = \{x \in X \mid x \geqslant 10\}$, we have $H(P) = \varnothing$.

2. $H(P) = \varnothing$, but $x \in X$ exists such that $x \gtrsim y$ for every $y \in P$. This point is said to be an upper bound of P and the set P is *bounded* (with respect to the relation $\gtrsim$).

Example 2.4.7

Take X and $\gtrsim$ as in the preceding example. For $P = \{x \in X \mid 0 \leqslant x < 7\}$, $H(P) = \varnothing$. The numbers 7, 10, 171 etc. are upper bounds of P and 7 is a lowest upper bound, but it is not a point of P.

Remark

If $H(P) = \varnothing$, the set P contains an infinite series of points $x_1, x_2, \ldots$, such that we have $x_2 \succ x_1, x_3 \succ x_2, \ldots$, while no point in P is preferred to all points of the series. In example 2.4.7, the series 6.9, 6.99, 6.999 etc. satisfies this condition.

$H(P) = \varnothing$ can only occur if P is an infinite set, since finite completely preordered sets always have a greatest element. Since in economics infinite sets are always approximations of finite sets, it is desirable to construct economic theories in such a way that the above cases are excluded.

2.5 UTILITY FUNCTIONS

A very important analytical instrument in choice theory is the utility function. 'Utility function' is a generally accepted term in choice theory for the mathematical concept 'order preserving function'.

Definition 2.5.1

If X is a set, completely ordered by a binary relation $\gtrsim$, then a mapping u: $X \rightarrow R$ is said to be an order-preserving function, if

$$u(x) > u(y) \Leftrightarrow x \succ y$$
$$u(x) = u(y) \Leftrightarrow x \sim y$$

If X is a choice space and $\gtrsim$ a preference relation, the mapping u is called a utility function and it associates with every point of X a real number, such that a point that is better than a second gets a higher number, while equivalent points get the same number. The number $u(x)$ is called a utility or a utility index. In a subset $P \subset X$ the best element has the highest utility.
An order-preserving function is not uniquely determined, since from one order-preserving function many others can be derived: every non decreasing transformation gives a new order-preserving function.

Theorem 2.5.2

If u: $X \to R$ is an order-preserving function and the function f: $R \to R$ satisfies

$$f(u) > f(u') \Leftrightarrow u > u' \qquad (u, u' \in R)$$

then

$$v(x) = f(u(x))$$

is also an order-preserving function.

Proof

Let $x \in X, y \in X$, now

$$x \succ y \Leftrightarrow u(x) > u(y) \Leftrightarrow v(x) = f(u(x)) > f(u(y)) = v(y)$$
$$x \sim y \Leftrightarrow u(x) = u(y) \Leftrightarrow v(x) = f(u(x)) = f(u(y)) = v(y)$$

Hence, a utility function only indicates order and it does not measure intensities. (It constitutes an 'ordinal scale'.[8]) This means that the utility difference $u(x) - u(y)$ has no significance. If by the preordering

$x \succ y$ and $z \succ v$,

the question whether x is more preferred to y than z to v, cannot be answered. If we have

$$u(x) - u(y) > u(z) - u(v)$$

a transformation of the utility function can converse this result. (For utility functions that do measure intensities, only linear transformations are allowed. In that case the utility constitutes an 'interval scale'.)

Remark

Order-preserving functions are not only applied in choice theory, but also streetnumbers, intelligence-quotients, production functions, index numbers etc. are order-preserving functions. A price-index number measures price movements. It is a mapping f: $R^m \to R$, such that

$$I = f(p_1, p_2, \ldots, p_m)$$

8. See STEVENS.

where p_i are the prices of m commodities. For a Laspeyres index, we have

$$I = \frac{\Sigma\, p_i x_i^0}{\Sigma\, p_i^0 x_i^0}$$

where p_i^0 and x_i^0 are prices and quantities of the commodities in a base period. Obviously, an index number is meant to be an interval scale.

3. Choice models

3.1 INTRODUCTION

In this chapter two related axiomatic choice models are considered, which are supposed to be a description of the behaviour of some individual.[1]
The models are based on four primitive concepts. These will be introduced in the next section. In section 3.3, a set of axioms will be given of a model that is called '*preference model*', since it is primarily based on a preference relation. In sections 3.4 and 3.5 this model is developed and its meaning is discussed in section 3.6. By a new set of axioms, another choice model is defined in section 7, and since this model is primarily based on a choice function, it is called '*choice function model*'.
In section 3.8 it is shown, that the axioms of the preference model can be derived as theorems from the choice function model. In section 3.9 some modifications of the choice function model are considered and the last section contains a summary of the connections between the models.
Both models are abstract, their preliminary concepts being very general. In chapters 5 and 6 two models are discussed, that are special cases of the models of the present chapter, and that have a narrower interpretation.

1. This chapter is in some way a generalisation of an article by Arrow (ARROW (1959)). (See also UZAWA (1956)) In this article it was assumed that a choice function is defined for every finite subset of the choice space. This restriction is not made here. Methodologically, our approach is somewhat different, because a preference relation and a choice function are introduced independently and associated by an axiom.

3.2 PRIMITIVE CONCEPTS

The primitive concepts of the models are the following:

an abstract set X (choice space)
a binary relation $\succsim$ on X (preference relation)
a set $\mathscr{P}$ of subsets of X (set of choice sets)
a correspondence $K\colon \mathscr{P} \to X$ (choice function)

The *choice space* was considered in 2.1. The elements of X are results (see 1.1) or probability distributions of results, that might be elements of a choice set. The concrete interpretation depends upon the problem at hand. The *preference relation* is a binary relation that can hold between two results and $\succsim$ is read 'preferred to' etc. (see 2.2). From $\succsim$ are derived a strict preference relation and an equivalence relation (see (2.2.3)).

Definition 3.2.1

If $x, y \in X$,

$$x \succ y \Leftrightarrow x \succsim y \wedge y \not\succsim x$$
$$x \sim y \Leftrightarrow x \succsim y \wedge y \succsim x$$

The choice set (see 2.1) is the set of results from which an individual has to choose one and only one element in a given situation. The set $\mathscr{P}$ contains all subsets of X that can possibly be choice sets for the individual, i.e. a situation can arise in which he has to choose from such a set. The extent of $\mathscr{P}$ depends on the nature of the problem. $\mathscr{P}$ is a subset of $\mathscr{A}$ (definition 2.4.5). $\mathscr{P}$ may coincide with $\mathscr{A}$; in that case any subset of X can be a choice set, but in many cases only certain subsets of X can be choice sets.

Example 3.2.2

Let X be a set of investment projects of a producer. A choice set contains projects he can realise in a given situation. Now in any situation he can choose for not investing, hence the result of this choice must always be an element of the choice set.

If $P \in \mathscr{P}$, the individual must be able to choose, i.e. decide which element or elements of P are suitable for choice. These suitable elements will be called the *eligible* elements of P. If P contains one eligible element, this will automatically be chosen, if there are more, one of these is arbitrarily chosen. The *choice function* $K: \mathscr{P} \to X$ is a *correspondence* that associates with every choice set its eligible elements.[2] Obviously

$$K(P) \subset P$$

and this property must be an axiom or be implied by the axioms.
Note that the preference relation reflects the individual's *opinions*, which are not directly observable, whereas the choice function reflects his *behaviour* and this can be observed in principle.

3.3 THE AXIOMS OF THE PREFERENCE MODEL

The preference model is characterised by the following set of axioms:

$P1$ (transitivity)

$$x \gtrsim y \wedge y \gtrsim z \Rightarrow x \gtrsim z$$

$P2$ (completeness)

$$\forall x, y \in X: x \gtrsim y \vee y \gtrsim x$$

$P3$ (selection axiom)

$$\forall P \in \mathscr{P}: H(P) \neq \varnothing.$$

$P4$ (transition axiom)

$$\forall P \in \mathscr{P}: K(P) = H(P)$$

By $P1$ and $P2$ the preference relation is a complete preordering (definition 2.2.19). Consequently, the model could only be applied in cases where these axioms give an adequate description of the individual's opinions, (and e.g. not in the case considered in remark 2.2.29).
In definition 2.4.6, we introduced the correspondence H, that associates with every subset of X its maximal elements, and since $\gtrsim$ is a complete preordering, the sets $H(P)$ contain greatest elements of P (see theorem 2.4.3). In 2.4 we have also seen that non empty infinite subsets of X can exist such that $H(P) = \varnothing$. It seems reasonable to exclude this case for choice sets, because infinite sets are mostly approximations of finite sets (and because an individual has to choose). Therefore axiom $P3$

2. The word 'function' is not correct, but it is generally used for this concept. Note that K is a mapping from $\mathscr{P}$ into $\mathscr{A}$.

requires that every choice set has a best element, hence only elements of $\mathscr{A}$ of this type can be selected as choice sets.
Axiom $P4$ requires that the eligible elements of a choice set are its best elements, hence

$$K(P) = \{x | y \in P \Rightarrow x \gtrsim y\}$$

and thus connects the preference relation and the choice function.[3] Hence, it is required that the individual *behaves in accordance with his opinions*. This means especially that $\gtrsim$ gets an operational meaning for two elements sets that are in $\mathscr{P}$. We then have:

$$x \gtrsim y \Rightarrow \{x\} = K(\{x, y\})$$
$$x \sim y \Leftrightarrow \{x, y\} = K(\{x, y\})$$

Applying $P4$, from $\gtrsim$ can be derived the set of eligible elements of every choice set and these elements must be at least as good as any other element of $P \in \mathscr{P}$, if they are compared separately.

Example

Let $X = \{a, b, c, d\}$ and $a \succ b \succ c \succ d$. It would be in contradiction with $P4$ if
$K(\{a, b, c\}) = \{b\}, K(\{b, c, d\}) = \{c\}$ etc.

In the preceding section it was mentioned that $K(P) \subset P$. This property is implied by the axioms.

Theorem 3.3.1

$$\forall P \in \mathscr{P}: K(P) \subset P$$

Proof

By $P4$, $K(P) = H(P)$ and by definition 2.4.6, $H(P) \subset P$. Hence, $K(P) \subset P$.

3.4 REVEALED PREFERENCE

In this section are considered some properties, a choice function must obey in a model characterised by axioms $P1$–$P4$.

3. In literature this axiom is seldom mentioned, mostly because of a different way of introducing concepts. In WOLD (1943) however appears an analogous formulation.

By the axioms every eligible element of a choice set must be at least as good as every other element of this set and better than every non-eligible element. Thus the preferences of the individual, reflected by $\gtrsim$, are partly revealed by the choice function: if $x \in K(P)$ and $y \in P$, this reveals that $x \gtrsim y$. Therefore we introduce a new binary relation for this case, denoted by R; $x\mathsf{R}y$ means that x is revealed preferred to y (or revealed at least as good as y). From R are derived two other relations P and I (in the same way as $\succ$ and $\sim$ were deduced from $\gtrsim$) and we shall show that it is allowed to read these 'revealed better' or 'revealed strictly preferred' and 'revealed equivalent'.

Definition 3.4.1

$$x\mathsf{R}y \Leftrightarrow [\exists P \in \mathscr{P}: x \in K(P) \wedge y \in P] \vee (x = y)$$

and

$$x\mathsf{I}y \Leftrightarrow x\mathsf{R}y \wedge y\mathsf{R}x$$
$$x\mathsf{P}y \Leftrightarrow x\mathsf{R}y \wedge y\not\mathsf{R}x$$

From this definition it follows:

Property 3.4.2

$$x\mathsf{R}y \Leftrightarrow (x\mathsf{I}y \vee x\mathsf{P}y) \wedge (x\not\mathsf{I}y \vee x\not\mathsf{P}y) \quad \text{(a)}$$
$$x\mathsf{I}y \Leftrightarrow y\mathsf{I}x \quad \text{(b)}$$
$$x\mathsf{R}y \Rightarrow y\not\mathsf{P}x \quad \text{(c)}$$
$$\forall x \in X: x\mathsf{I}x \quad \text{(d)}$$

Remark 3.4.3

In the definition of R is added the case $x = y$, to guarantee reflexivity (d). If this would not be done, $x\mathsf{I}x$ would only hold if x would be an eligible element of at least one choice set.

Definition 3.4.1. implies $x\mathsf{I}y$, if two sets P and Q exist, both containing x and y, while x is eligible from P and y from Q

$$(\exists P: x \in K(P) \wedge y \in P) \wedge (\exists Q: x \in Q \wedge y \in K(Q))$$

This might happen in two cases:
a. if $x, y \in K(P)$ (Now $Q = P$, see figure 3.4.4a)

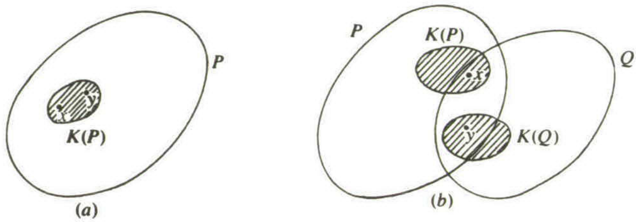


fig. 3.4.4

b. if $x \in K(P)$, $y \in P$, but $y \notin K(P)$, and $y \in K(Q)$, $x \in Q$, but $x \notin K(Q)$. (see figure 3.4.4b).

It is shown that case (*b*) is excluded in the present model: it is impossible that x is eligible from a choice set P, whereas y is a non eligible point of P, while y is eligible from another choice set Q, that also contains x. This property is known as 'the *weak* property of revealed preference' (generally introduced as an axiom[4]).

Theorem 3.4.5 (weak theorem of revealed preference)

$$x \in K(P) \wedge y \in P \wedge y \notin K(P) \Rightarrow \nexists Q: x \in Q \wedge y \in K(Q)$$

Proof

x must be a best element of P and y is not, so $x \gtrsim y$ and $y \not\gtrsim x$. If $x \in Q$ and $y \in K(Q)$, then $y \gtrsim x$, which is a contradiction.

This theorem implies that $x \mathrm{I} y$ holds if and only if x and y appear as eligible elements in the same choice set, and that we have $x \mathrm{P} y$ if x is eligible from a choice set that also contains the non eligible element y.

Remark 3.4.6

Usually the statements of theorem 3.4.7 are used as definitions of I and P. Our definition then follows as a theorem.

Theorem 3.4.7

$$x \mathrm{I} y \Leftrightarrow (\exists P \in \mathscr{P}: x, y \in K(P)) \vee x = y \quad \text{(a)}$$
$$x \mathrm{P} y \Leftrightarrow \exists P \in \mathscr{P}: x \in K(P) \wedge y \in P \wedge y \notin K(P) \quad \text{(b)}$$

4. See e.g. ARROW (1959). The weak axiom of revealed preference was first introduced by Samuelson (SAMUELSON (1938), p. 65).

Proof

b. $\Rightarrow$ by definition 3.4.1.
$\Leftarrow$ by definition 3.4.1, we have $x\mathsf{R}y$; by theorem 3.4.5, $y\mathsf{R}x$ is excluded.

a. $\Rightarrow$ by definition; $\Leftarrow y \notin K(P)$ would be a contradiction with (b).

Obviously revealed preference implies preference.

Theorem 3.4.8

$$x\mathsf{R}y \Rightarrow x \succsim y$$
$$x\mathsf{I}y \Rightarrow x \sim y$$
$$x\mathsf{P}y \Rightarrow x \succ y$$

Proof

If $x \in K(P)$ and $y \in P$, we must have $x \succsim y$, since by $P4$, x is a best element. If $y \notin K(P)$, y is not a best element, hence $x \succ y$. If both x and y are in $K(P)$, we have $x \sim y$.

The converse of theorem 3.4.8 is not true. The relation $x\mathsf{R}y$ only holds if it can be derived from the choice function and it is possible that two points x and y never appear simultaneously in a choice set. Hence R needs not to be complete. R is generally not transitive either; we may have $x\mathsf{R}z$ and $z\mathsf{R}y$, but not $x\mathsf{R}y$: x is not eligible from a choice set that contains y, but P and Q exist, such that $x \in K(P)$, $z \in P$ and $z \in K(Q)$, $y \in Q$. (see fig. 3.4.9).

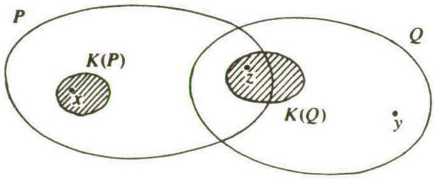


fig. 3.4.9

In this case we shall say that x is revealed preferred to y in two steps, denoted $x\mathsf{R}^2y$. In general, x is revealed preferred to y in k steps, if, with the help of $k-1$ points of X, a chain of preferences between x and y can be constructed, where k is a natural number ($k \in N$ and $N = \{1, 2, 3, \ldots\}$). This is written $x\mathsf{R}^ky$.

Definition 3.4.10

For k $\in N$

$$x\mathsf{R}^k y \Leftrightarrow \exists z_1, z_2, \ldots, z_{k-1}: x\mathsf{R}z_1, z_1\mathsf{R}z_2, \ldots, z_{k-1}\mathsf{R}y$$

Obviously $x\mathsf{R}^k y$ implies $x \gtrsim y$. ($x\mathsf{P}^k y$ and $x\mathsf{I}^k y$ can also be defined.) The relation $x\mathsf{R}^k y$ is generally not transitive for any k: $x\mathsf{R}^k z$ and $z\mathsf{R}^k y$ only imply $x\mathsf{R}^{2k} y$. Therefore we introduce a third revealed preference concept for elements between which exists a chain of preferences of arbitrary length; this relation is denoted $\bar{\mathsf{R}}$, with derived relations $\bar{\mathsf{P}}$ and $\bar{\mathsf{I}}$. $\bar{\mathsf{R}}$ is called an *indirect* revealed preference relation. (Therefore R will henceforward be called a direct revealed preference relation.)

Definition 3.4.11

$$x\bar{\mathsf{R}}y \Leftrightarrow \exists k \in N: x\mathsf{R}^k y$$

and

$$x\bar{\mathsf{I}}y \Leftrightarrow x\bar{\mathsf{R}}y \wedge y\bar{\mathsf{R}}x$$
$$x\bar{\mathsf{P}}y \Leftrightarrow x\bar{\mathsf{R}}y \wedge y\not{\bar{\mathsf{R}}}x$$

$x\bar{\mathsf{R}}y$ is read 'x is indirectly revealed preferred to y'. In theorem 3.4.17 below it will be shown that it is permitted to read $x\bar{\mathsf{P}}y$ 'x is revealed better' (strictly preferred) and to read $x\bar{\mathsf{I}}y$, 'x is revealed equivalent to y'. From this definition it follows

Property 3.4.12

$$x\bar{\mathsf{R}}y \Leftrightarrow (x\bar{\mathsf{I}}y \vee x\bar{\mathsf{P}}y) \wedge (x\not{\bar{\mathsf{I}}}y \vee x\not{\bar{\mathsf{P}}}y) \quad \text{(a)}$$
$$x\bar{\mathsf{I}}y \Leftrightarrow y\bar{\mathsf{I}}x \quad \text{(b)}$$
$$x\bar{\mathsf{R}}y \Rightarrow y\not{\bar{\mathsf{P}}}x \quad \text{(c)}$$
$$x\mathsf{R}y \Rightarrow x\bar{\mathsf{R}}y \quad \text{(d)}$$

Remark 3.4.13

Note that from the *definitions* can*not* be deduced:

$x\mathsf{I}y \Rightarrow x\bar{\mathsf{I}}y$ or $x\mathsf{P}y \Rightarrow x\bar{\mathsf{P}}y$

This is due to the way in which the direct revealed preference relations were defined (see remark 3.4.6). The above statement is implied by *theorem* 3.4.15.

Definition 3.4.11 also implies, that $\bar{\mathsf{R}}$ is a partial preordering.

Property 3.4.14

$x\bar{\mathsf{R}}y \wedge y\bar{\mathsf{R}}z \Rightarrow x\bar{\mathsf{R}}z$ (transitivity) (a)

$\forall x \in X: x\bar{\mathsf{R}}x$ (reflexivity) (b)

Proof

a. By definitions 3.4.10 and 3.4.11, $k \in N$ and $l \in N$ must exist, such that $x\mathsf{R}^k y$ and $y\mathsf{R}^l z$; hence s_i and t_j exist such that

$x\mathsf{R}s_1 \wedge s_1\mathsf{R}s_2 \wedge \ldots \wedge s_{k-1}\mathsf{R}y$

and

$y\mathsf{R}t_1 \wedge t_1\mathsf{R}t_2 \wedge \ldots \wedge t_{l-1}\mathsf{R}z$

This means

$x\mathsf{R}^{k+l}y$

Hence $x\bar{\mathsf{R}}y$.

b. Since $\forall x$: $x\mathsf{R}x$, also $x\bar{\mathsf{R}}x$.

The interpretation of P and I, given above, is only allowed, if there cannot arise contradiction with R, which would occur, if $x\bar{\mathsf{R}}y$ and simultaneously yPx (x is indirectly revealed preferred to y and y is directly revealed strictly preferred to y).

Theorem 3.4.15 (strong theorem of revealed preference)

$x\bar{\mathsf{R}}y \Rightarrow y \not{P} x$

Proof

If $x\bar{\mathsf{R}}y$, for some $k \in N$, $x\mathsf{R}^k y$. Hence s_i exist, such that

$x\mathsf{R}s_1 \wedge s_1\mathsf{R}s_2 \wedge \ldots \wedge s_{k-1}\mathsf{R}y$

which implies, by theorem 3.4.8,

$x \gtrsim s_1 \wedge s_1 \gtrsim s_2 \wedge \ldots \wedge s_{k-1} \gtrsim y$

and thus by axiom $P1$, $x \gtrsim y$.
Now $y\mathsf{P}x$ would mean $y \succ x$, which is a contradiction.

This is generally known as the *strong* axiom of revealed preference.[5]

Remark 3.4.16

Due to our way of defining the direct revealed preference relation, the strong property of revealed preference *does not imply* the weak property. By the statement of theorem 3.4.15, the case of figure 3.4.4b is not excluded, since in that case only $x\mathsf{I}y$ holds.

The following theorem follows from theorem 3.4.15.

Theorem 3.4.17

$$x\bar{\mathsf{I}}y \Leftrightarrow \exists s_1, s_2, \ldots, s_{k-1}: x\mathsf{I}s_1 \wedge s_1\mathsf{I}s_2 \wedge \ldots \wedge s_{k-1}\mathsf{I}y \qquad \text{(a)}$$
$$x\bar{\mathsf{P}}y \Leftrightarrow \exists s_1, s_2, \ldots, s_{k-1}: x\mathsf{R}s_1 \wedge s_1\mathsf{R}s_2 \wedge \ldots \wedge s_{k-1}\mathsf{R}y \qquad \text{(b)}$$

and in the chain occurs at least one P.

Proof

a. $\Leftarrow$ follows from definition 3.4.11.
$\Rightarrow$ If $x\bar{\mathsf{I}}y$, then $k \in N$ and $l \in N$ exist, such that
$x\mathsf{R}^k y$ and $y\mathsf{R}^l x$
In both chains of preferences, the relation P cannot occur, otherwise a contradiction with theorem 3.4.15 would arise.
b. follows from (*a*) by applying property 3.4.12.

Theorem 3.4.18

$$x\bar{\mathsf{R}}y \Rightarrow x \succsim y$$
$$x\bar{\mathsf{I}}y \Rightarrow x \sim y$$
$$x\bar{\mathsf{P}}y \Rightarrow x \succ y$$

Proof

This is easily proved by applying theorem 3.4.8 and definition 3.4.11 to the preceding theorem.

Hence, if the axioms are valid, the preference relation can *partly* be reconstructed with the help of a known choice function, defined on a set of choice sets.

5. This property was first established by VILLE (1945) and HOUTHAKKER (1948).

Another property, which can be deduced from the model and which has some intuitive appeal says that, if we form a new choice set by removing some elements from a choice set, but not all eligible elements, the remaining eligible elements are the eligible elements in the new set, provided that the new set is in $\mathscr{P}$. This is especially true when only non eligible elements are removed; in that case the set of eligible elements does not change. This property is called 'independence of irrelevant alternatives.[6] (see figure 3.4.19).

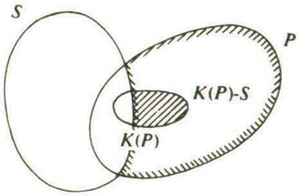


fig. 3.4.19

Theorem 3.4.20

$$S \subset X \wedge P,(P-S) \in \mathscr{P} \wedge K(P)-S \neq \varnothing \Rightarrow$$
$$K(P-S) = K(P) - S$$

Proof

1. $K(P-S) \supset K(P)-S$:
If $x_0 \in K(P)-S$, then $y \in P \Rightarrow x_0 \gtrsim y$, and so also $y \in (P-S) \Rightarrow x_0 \gtrsim y$, which means by definition, that x_0 is a best element of $(P-S)$, or $x_0 \in K(P-S)$.
2. $K(P-S) \subset K(P)-S$:
Let $y_0 \in K(P-S)$; $x_0 \in K(P)-S$ exists, hence $x_0 \in (P-S)$, which means $y_0 \gtrsim x_0$. Now $y_0 \in K(P)$ and therefore $y_0 \in K(P)-S$.

Remark 3.4.21

This theorem is also a direct result of the weak theorem of revealed preference. The converse is not generally true. Theorem 3.4.5 can only be deduced from theorem 3.4.20, if $\mathscr{P}$ satisfies certain conditions (see section 3.9).

6. See e.g. ARROW (1951) and LUCE (1959).

In this section, we shall define a number of binary relations on the set of choice sets $\mathscr{P}$. These relations are similar to the relations $\gtrsim$, R, R^k and $\bar{\mathsf{R}}$ on the choice space X and in the preference model they have roughly the same properties. The relations, denoted by $\gtrsim^*$, R^*, R^{*k} and $\bar{\mathsf{R}}^*$ are defined in terms of the preference relation and the choice function.
We say that one choice set is at least as favourable as another, if the best elements of the first set are at least as good as the best elements of the second one.[7]

Definition 3.5.1

If $P, Q \in \mathscr{P}$

$$P \gtrsim^* Q \Leftrightarrow \exists x, \exists y \in X: x \in K(P) \wedge y \in K(Q) \wedge x \gtrsim y$$

and

$$P \sim^* Q \Leftrightarrow P \gtrsim^* Q \wedge Q \gtrsim^* P$$
$$P \succ^* Q \Leftrightarrow P \gtrsim^* Q \wedge Q \not\gtrsim^* P$$

Now $\sim^*$ and $\succ^*$ can be read 'equally favourable as' and 'more favourable than'. Clearly, these relations merely reflect opinions, choice between sets being impossible.
Since $\gtrsim$ on X is a complete preordering by $P1$ and $P2$, and since $K(P)$ contains all best elements, this definition is equivalent to

$$P \gtrsim^* Q \Leftrightarrow (x \in K(P) \wedge y \in K(Q) \Rightarrow x \gtrsim y).$$

and hence for $\sim^*$ and $\succ^*$ the following theorem holds.

Theorem 3.5.2

$$P \sim^* Q \Leftrightarrow \exists x, \exists y \in X: x \in K(P) \wedge y \in K(Q) \wedge x \sim y$$
$$P \succ^* Q \Leftrightarrow \exists x, \exists y \in X: x \in K(P) \wedge y \in K(Q) \wedge x \succ y$$

Hence, the choice sets are ordered as their best elements; therefore $\gtrsim^*$ is a *complete preordering*, as $\gtrsim$ is.

7. This is an interpretation of the relation of definition 3.5.4. This interpretation does not consider the possibility that the individual, comparing choice sets, also takes into account their extent.

Theorem 3.5.3

$$P \gtrsim^* Q \wedge Q \gtrsim^* R \Rightarrow P \gtrsim^* R$$
$$\forall P, \forall Q \in \mathscr{P}: P \gtrsim^* Q \vee Q \gtrsim^* P$$

Like preference, favourability can be revealed by the choice function. We say that the choice set P is directly *revealed at least as favourable as* Q, if P contains eligible elements of Q.

Definition 3.5.4

If $P, Q \in \mathscr{P}$

$$P\mathsf{R}^*Q \Leftrightarrow P \cap K(Q) \neq \varnothing$$

and

$$P\mathsf{I}^*Q \Leftrightarrow P\mathsf{R}^*Q \wedge Q\mathsf{R}^*P$$
$$P\mathsf{P}^*Q \Leftrightarrow P\mathsf{R}^*Q \wedge Q\not\mathsf{R}^*P$$

This definition parallels definition 3.4.1 of direct revealed preference. I^* (P^*) can be read 'revealed equally (more) favourable as (than) Q and they hold if P and Q contain eligible elements of each other (P contains eligible elements of Q, but Q does not contain eligible elements of P). From the definition can be deduced the following set of properties.

Property 3.5.5

$$P\mathsf{R}^*Q \Leftrightarrow (P\mathsf{P}^*Q \vee P\mathsf{I}^*Q) \wedge (P\not\mathsf{P}^*Q \vee P\not\mathsf{I}^*Q) \quad \text{(a)}$$
$$P\mathsf{I}^*Q \Leftrightarrow Q\mathsf{I}^*P \quad \text{(b)}$$
$$P\mathsf{R}^*Q \wedge P \subset S \Rightarrow S\mathsf{R}^*Q \quad \text{(c)}$$
$$P\mathsf{R}^*Q \Rightarrow Q\not\mathsf{P}^*P \quad \text{(d)}$$
$$\forall P \in \mathscr{P}: P\mathsf{I}^*P \quad \text{(e)}$$

If $P\mathsf{I}^*Q$, by definition each set contains eligible elements of the other. Now eligible elements which one set contains of the other, are eligible in both sets. This is a direct consequence of the weak theorem of revealed preference of which the following theorem is another version.

Theorem 3.5.6.

$$P\mathsf{I}^*Q \Leftrightarrow P \cap K(Q) = K(P) \cap Q \neq \varnothing.$$

Proof

$\Leftarrow$ is true by definition.
$\Rightarrow$ Let $x \in K(P) \cap Q$ and $y \in P \cap K(Q)$. Suppose $y \notin K(P) \cap Q$; now $x \in K(P)$, $y \in P$, $y \notin K(P)$ and $x \in Q$ and $y \in K(Q)$, which contradicts theorem 3.4.5. So $y \in K(P) \cap Q$. In the same way it is shown that $x \in P \cap K(Q)$.

In figure 3.5.7 this theorem is illustrated (see also figure 3.4.4)

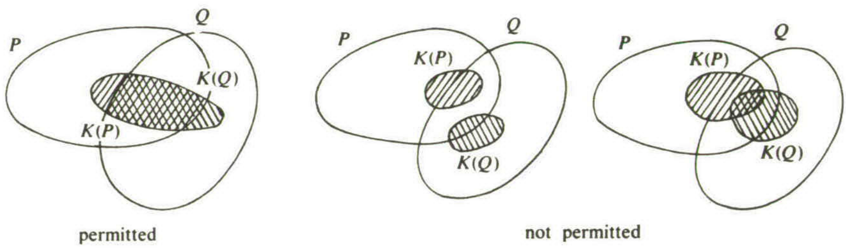


fig. 3.5.7

It is not allowed that a point of $P \cap Q$ is eligible from P and not from Q, whereas another point of the intersection is eligible from Q.

Example 3.5.8

Let $X = \{$a(pple), b(anana), g(rapes), o(range)$\}$.
Suppose $\{o\} = K(\{a, g, o\})$.
It is impossible that $a \in K(\{a, b, o\})$, but it is permitted that

$$\{o\} = K(\{a, b, o\}) \text{ hence } \{a, b, o\}\mathsf{I}^*\{a, g, o\}$$
$$\{b\} = K(\{a, b, o\}) \text{ hence } \{a, b, o\}\mathsf{P}^*\{a, g, o\}$$
$$\{b, o\} = K(\{a, b, o\}) \text{ hence } \{a, b, o\}\mathsf{I}^*\{a, g, o\}$$

Revealed favourability implies favourability:

Theorem 3.5.9

$$P\mathsf{R}^*Q \Rightarrow P \gtrsim^* Q \qquad \text{(a)}$$
$$P\mathsf{I}^*Q \Rightarrow P \sim^* Q \qquad \text{(b)}$$
$$P\mathsf{P}^*Q \Rightarrow P \succ^* Q \qquad \text{(c)}$$

Proof

a. By definitions 3.5.4 and 3.4.7:

$P\mathsf{R}^*Q \Rightarrow \exists x, \exists y: x \in K(P) \wedge y \in K(Q) \wedge x\mathsf{R}y$

hence (theorem 3.4.8)

$\exists x, \exists y: x \in K(P) \wedge y \in K(Q) \wedge x \gtrsim y$

hence

$P \gtrsim^* Q$

b. $P\mathsf{I}^*Q \Rightarrow \exists x, \exists y: x \in K(P) \wedge y \in K(Q) \wedge x\mathsf{I}y$

hence

$\exists x, \exists y: x \in K(P) \wedge y \in K(Q) \wedge x \sim y$

c. is implied by (b) (property 3.5.5).

No more than R, R* is a preordering. It is neither transitive (see figure 3.5.10a) nor complete (see figure 3.5.10b).

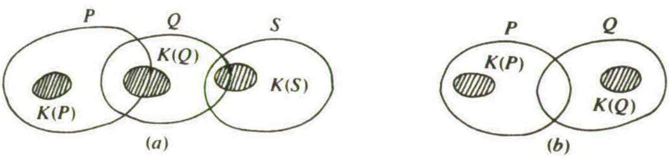


fig. 3.5.10

Therefore, we define on $\mathscr{P}$ a stepwise favourability relation and an indirect favourability relation, similar to R^k and $\bar{\mathsf{R}}$ on X (definitions 3.4.10 and 3.4.11).

Definition 3.5.11

For $k \in N$:

$$P\mathsf{R}^{*k}Q \Leftrightarrow \exists S_1, S_2, \ldots, S_{k-1}: P\mathsf{R}^*S_1, S_1\mathsf{R}^*S_2, \ldots, S_{k-1}\mathsf{R}^*Q$$

Hence, every set must contain eligible elements of the next set in the chain. R^{*k} is read 'revealed at least as favourable in k steps', whereas $\bar{\mathsf{R}}^*$ is read 'indirectly revealed at least as favourable'.

Definition 3.5.12

$$P\bar{\mathsf{R}}^*Q \Leftrightarrow \exists k \in N: P\mathsf{R}^{*k}Q$$

and

$$P\bar{\mathrm{I}}^*Q \Leftrightarrow P\bar{\mathrm{R}}^*Q \wedge Q\bar{\mathrm{R}}^*P$$
$$P\bar{\mathrm{P}}^*Q \Leftrightarrow P\bar{\mathrm{R}}^*Q \wedge Q\not\bar{\mathrm{R}}^*P$$

From this definition can be deduced:

Property 3.5.13

$$P\bar{\mathrm{R}}^*Q \Rightarrow (P\bar{\mathrm{P}}^*Q \vee P\bar{\mathrm{I}}^*Q) \wedge (P\not\bar{\mathrm{P}}^*Q \vee P\not\bar{\mathrm{I}}^*Q) \qquad \text{(a)}$$
$$P\bar{\mathrm{I}}^*Q \Leftrightarrow Q\bar{\mathrm{I}}^*P \qquad \text{(b)}$$
$$P\bar{\mathrm{R}}^*Q \wedge P \subset S \in \mathscr{P} \Rightarrow S\bar{\mathrm{R}}^*Q \qquad \text{(c)}$$
$$P\bar{\mathrm{R}}^*Q \Rightarrow Q\not\bar{\mathrm{P}}^*P \qquad \text{(d)}$$
$$\forall P \in \mathscr{P}: P\bar{\mathrm{I}}^*P \qquad \text{(e)}$$
$$P\mathrm{R}^*Q \Rightarrow P\bar{\mathrm{R}}^*Q \qquad \text{(f)}$$

Like $\bar{\mathrm{R}}$, $\bar{\mathrm{R}}^*$ is a complete preordering (see property 3.4.14).

Property 3.5.14

$$P\bar{\mathrm{R}}^*Q \wedge Q\bar{\mathrm{R}}^*S \Rightarrow P\bar{\mathrm{R}}^*S \qquad \text{(a)}$$
$$\forall P \in \mathscr{P}: P\bar{\mathrm{R}}^*P \qquad \text{(b)}$$

Proof

a. $k \in N$ and $l \in N$ must exist, such that $P\mathrm{R}^{*k}Q$ and $Q\mathrm{R}^{*l}S$, hence $P\mathrm{R}^{*k+l}S$.

The binary relations $\bar{\mathrm{R}}$ and $\bar{\mathrm{R}}^*$ are related by the following expression:

Property 3.5.15

$$x\bar{\mathrm{R}}y \Leftrightarrow \exists P, Q \in \mathscr{P}: x \in K(P) \wedge y \in Q \wedge P\bar{\mathrm{R}}^*Q$$

Proof

By definitions 3.4.10 and 3.4.11, we have $x\bar{\mathrm{R}}y \Leftrightarrow \exists k \in N, \exists s_i \in X$: $x\mathrm{R}s_1 \wedge s_1\mathrm{R}s_2 \wedge \ldots \wedge S_{k-1}\mathrm{R}y$ and this implies the existence of choice sets S_i $(i = 1, 2, \ldots, k-1)$, such that $x \in K(P)$, $s_1 \in P \cap K(S_1)$, $s_2 \in S_1 \cap K(S_2), \ldots, s_{k-1} \in S_{k-2} \cap K(S_{k-1})$, $y \in S_{k-1}$; hence

$$P\mathrm{R}^*S_1 \wedge S_1\mathrm{R}^*S_2 \wedge \ldots \wedge S_{k-1}\mathrm{R}^*Q$$

and this implies $P\mathsf{R}^{*k}Q$ and hence $P\bar{\mathsf{R}}^{*}Q$. The converse is proved in a similar way.

In certain cases $\bar{\mathsf{R}}^{*}$ is complete. This is true e.g. if the union of every pair of choice sets is again a choice set. (This is a sufficient, not a necessary condition.)

Theorem 3.5.16

$$\forall P, \forall Q \in \mathscr{P}: (P \cup Q) \in \mathscr{P} \Rightarrow \forall P, \forall Q \in \mathscr{P}: P\bar{\mathsf{R}}^{*}Q \vee Q\bar{\mathsf{R}}^{*}P$$

Proof

By definition 3.5.4.

$(P \cup Q)\mathsf{R}^{*}P$ and $(P \cup Q)\mathsf{R}^{*}Q$

Since $K(P \cup Q) \subset (P \cup Q)$, of (a) and (b) at least one is true:

(a) $K(P \cup Q) \cap Q \neq \varnothing$, hence $Q\mathsf{R}^{*}(P \cup Q)$,
(b) $K(P \cup Q) \cap P \neq \varnothing$, hence $P\mathsf{R}^{*}(P \cup Q)$.
If (a), by $Q\mathsf{R}^{*}(P \cup Q)$ and $(P \cup Q)\mathsf{R}^{*}P$, $Q\bar{\mathsf{R}}^{*}P$.
If (b), $P\bar{\mathsf{R}}^{*}Q$.

The condition of the theorem is satisfied, if

$$\mathscr{P} = \{P \in \mathscr{A} \mid H(P) \neq \varnothing\}$$

i.e. if $\mathscr{P}$ contains all subsets of X that have a best (greatest) element. In general, $\bar{\mathsf{R}}^{*}$ is complete, if the set $\mathscr{P}$ is sufficiently extensive and this depends also on the choice function.

Example 3.5.17

Let $X = \{a, b, c, d\}$.
a. $\mathscr{P} = \{\{a, b\}, \{b, c\}, \{c, d\}\}$
$K(\{a, b\}) = \{a\}$, $K(\{b, c\}) = \{c\}$ and $K(\{c, d\}) = \{c\}$.
Now $\{a, b\}$ and $\{c, d\}$ are not comparable, and $\{b, c\}\bar{\mathsf{I}}^{*}\{c, d\}$.
b. Let $\mathscr{P}$ be defined as in (a).
$K(\{a, b\}) = \{a\}$; $K(\{b, c\}) = \{b\}$ and $K(\{c, d\}) = \{c\}$.
Now the preordering is complete, however the condition of theorem 3.5.16 is not satisfied.
$\{a, b\}\mathsf{P}^{*}\{b, c\}$, $\{b, c\}\mathsf{P}^{*}\{c, d\}$
and by transitivity of $\bar{\mathsf{P}}^{*}$ (see below),
$\{a, b\}\bar{\mathsf{P}}^{*}\{c, d\}$.

For $\bar{\mathsf{R}}^*$ as for $\bar{\mathsf{R}}$, a 'revealed preference theorem' can be derived, which excludes contradiction between $\bar{\mathsf{R}}^*$ and P^* (see theorem 3.4.15).

Theorem 3.5.18

$$P\bar{\mathsf{R}}^*Q \Rightarrow Q\not{\mathsf{P}}^*P$$

Proof

If $P\bar{\mathsf{R}}^*Q$, $k \in N$ exists such that

$P\mathsf{R}^*S_1 \wedge S_1\mathsf{R}^*S_2 \ldots \wedge S_{k-1}\mathsf{R}^*Q$

By theorem 3.5.9 this implies

$P \gtrsim^* S_1 \wedge S_1 \gtrsim^* S_2 \ldots \wedge S_{k-1} \gtrsim^* Q$

and hence $P \gtrsim^* Q$.

But $Q\mathsf{P}^*P$ would imply $Q \succ^* P$ and this is a contradiction.

Now it follows:

Theorem 3.5.19

$$P\bar{\mathsf{I}}^*Q \Leftrightarrow \exists S_1, S_2, \ldots, S_n\colon P\mathsf{I}^*S_1 \wedge S_1\mathsf{I}^*S_2 \ldots \wedge S_n\mathsf{I}^*Q.$$

$$P\bar{\mathsf{P}}^*Q \Leftrightarrow \exists S_1, S_2, \ldots, S_n\colon P\mathsf{R}^*S_1 \wedge S_1\mathsf{R}^*S_2 \ldots \wedge S_n\mathsf{R}^*Q$$

and in this chain appears at least one P^*.

Proof

See proof of theorem 3.4.17.

Obviously, revealed favourability implies favourability (see theorem 3.4.18).

Theorem 3.5.20

$$P\bar{\mathsf{R}}^*Q \Rightarrow P \gtrsim^* Q$$
$$P\bar{\mathsf{I}}^*Q \Rightarrow P \sim^* Q$$
$$P\bar{\mathsf{P}}^*Q \Rightarrow P \succ^* Q$$

3.6 THE LOGICAL SIGNIFICANCE OF THE PREFERENCE MODEL

The significance of the theory developed in the previous sections can be summarized in the following statements:

1. If a preference relation on some choice space is a complete preordering and the selection- and transition-axioms are true, then all theorems derived in the preceding sections are true statements.
2. If a preference relation on a choice space is specified, and it happens to be a complete preordering, and also for a given set $\mathscr{P}$ the transition- and selection-axioms are known to be true, then the choice function can be derived, and all theorems of the preceding sections must be true for this function.
3. If both a preference relation on a set X and a choice function on a set of subsets of X, $\mathscr{P}$, are specified, then either all axioms are verified and all theorems are true, or some theorems are not true and at least one of the axioms is not true.
4. If it is known that all axioms must be true and the choice function is given for some $\mathscr{P}$, then for this choice function all theorems must be true statements, and by the revealed preference concept the preference relation can at least be reconstructed partially.

3.7 A CHOICE FUNCTION MODEL

The preference model is based on a binary relation that reflects the opinions of some individual. These opinions are never directly expressed, hence they are generally not observable. Therefore, it is hardly ever possible to find out if the axioms are true.

In some cases the choices from certain sets are observable. Therefore, it seems useful to base a choice model on a choice function. This model is called choice function model (also denoted model K).

We now start with the same primitive concepts as in the preference model (see section 3.2). However, we introduce a set of axioms that especially refer to the choice function. It is shown in the next section that this model implies the preference model.

All *definitions* of the preceding sections are maintained. The '*properties*' remain true, because they depend only on the definitions (especially the transitivity of $\bar{\mathsf{R}}$ and $\bar{\mathsf{R}}^*$).

The theorems of course are dropped, since they are based on the axioms. Some of the theorems are introduced as axioms in the new model, other theorems are now proved in the choice function model.

We shall show, that axioms $P1$–$P4$ can be derived as theorems in the choice function model; consequently, the choice function model contains *sufficient* conditions for the preference model.

Axioms of model K.

$K1$

$\forall P \in \mathscr{P}: K(P) \neq \varnothing$

$K2$

$\forall P \in \mathscr{P}: K(P) \subset P$

$K3$ (weak axiom of revealed preference)

$$\left.\begin{array}{ll} x \in K(P) & \wedge \\ y \in P & \wedge \\ y \notin K(P) & \end{array}\right\} \Rightarrow \nexists Q: y \in K(Q) \wedge x \in Q$$

$K4$ (strong axiom of revealed preference)

$P\bar{\mathsf{R}}^*Q \Rightarrow Q\bar{\not\mathsf{P}}^*P$

$K5$ (transition axiom)

$x\bar{\mathsf{R}}y \Rightarrow x \gtrsim y$
$x\bar{\mathsf{P}}y \Rightarrow x \succ y$

$K6$

$\forall P, Q \in \mathscr{P}: P\bar{\mathsf{R}}^*Q \vee Q\bar{\mathsf{R}}^*P$

$K7$

$\forall x \in X, \exists P \in \mathscr{P}: x \in K(P)$

The first 5 axioms are derived as theorems in the preference model: $K1$ follows from $P1$ and $P2$. $K2$, $K3$, $K4$ and $K5$ are identical respectively with theorems 3.3.1, 3.4.5, 3.5.18 and 3.4.18. Thus, they are necessary conditions for model P. Axioms $K6$ and $K7$ however do not follow from model P, hence they are not necessary.
The axioms can be interpreted independently.

- $K1$ From every choice set the individual is able to choose.
- $K2$ An eligible element is always in the choice set.
- $K3$ It is impossible for an element to be eligible from one set in the presence of a second element, which is not eligible from this set, but which is eligible from another set in the presence of the first element. (The case of fig. 3.7.1a is excluded.)
- $K4$ A choice set cannot be indirectly revealed at least as favourable as a second, whereas the second is directly revealed more favourable. (The case of fig. 3.7.1b is excluded.)

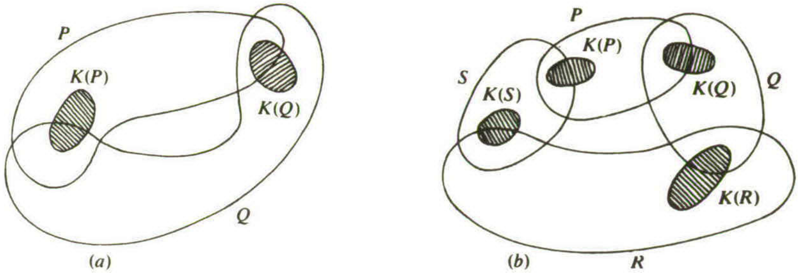


fig. 3.7.1

$K5$ An element that is *revealed* at least as good as (better than) a second, *is* at least as good as (better than) the second.

These axioms seem intuitively plausible.

The last two axioms cannot be made acceptable by intuitive reasoning. They may be valid or not.

$K6$ The set $\mathscr{P}$ is sufficiently extensive with respect to $K(P)$, to guarantee the relation $\bar{\mathsf{R}}^*$ to be complete.

$K7$ Every element of X is eligible from at least one choice set.

3.8 THE CONNECTION BETWEEN THE CHOICE FUNCTION MODEL AND THE PREFERENCE MODEL

We shall first show that axioms $P3$ and $P4$ can be deduced from $K1$–$K5$, without using $K6$ and $K7$.

The strong property of revealed preference also holds for $\bar{\mathsf{R}}$, and not only for $\bar{\mathsf{R}}^*$ as was stated in axiom $K4$. The statement of the next theorem is identical to that of theorem 3.4.15.

Theorem 3.8.1

$$x\bar{\mathsf{R}}y \Rightarrow y\not{\mathsf{P}}x$$

Proof

If $x\bar{\mathsf{R}}y$, by property 3.5.15, choice sets P and S exist, such that

$$x \in K(P) \wedge P\bar{\mathsf{R}}^*S \wedge y \in S$$

Suppose also $y\mathsf{P}x$; now by def. 3.4.1, Q must exist, such that

$$y \in K(Q) \wedge x \in Q \wedge x \notin K(Q)$$

which implies $Q\mathsf{R}^*P$, and by $K3$, also $Q\mathsf{P}^*P$. Since also $S\mathsf{R}^*Q$, we have

$P\bar{\mathsf{R}}^*S \wedge S\mathsf{R}^*Q \wedge Q\mathsf{P}^*P$

which contradicts $K4$.

If, by applying $K5$, the preference relation $\gtrsim$ is derived from $\bar{\mathsf{R}}$, and if then with the help of $\gtrsim$, the correspondence H is constructed, then $H(P)$ is identical to $K(P)$ for $P \in \mathscr{P}$. Thus, axiom $P4$ is implied by $K1$–$K5$.

Theorem 3.8.2

$\forall P \in \mathscr{P}: K(P) = H(P)$

Proof

1. $x \in K(P) \Rightarrow x \in H(P)$.
By $K2$, $x \in P$. By def. 3.4.1, $y \in P \Rightarrow x\mathsf{R}y$ and hence $x\bar{\mathsf{R}}y$, which implies $x \gtrsim y$. Hence $x \in H(P)$.
2. $y \notin K(P) \Rightarrow y \notin H(P)$
By $K1$, $K(P) \neq \varnothing$. Hence $x \in K(P)$ exists.
Let $y \in P$ and $y \notin K(P)$. Now $x \in H(P)$, by (1), and $x\mathsf{R}y$.
By $K3$, $y\mathsf{R}x$ is excluded and this means $x\mathsf{P}y$, by def. 3.4.1.
By theorem 3.8.1,
$x\mathsf{P}y \Rightarrow y\bar{\not{\mathsf{R}}}x$
and therefore by def. 3.4.11, $x\bar{\mathsf{P}}y$.
This implies $x \succ y$ and hence $y \notin H(P)$.

By applying theorem 3.8.2 to axiom $K1$, it follows immediately:

Theorem 3.8.3

$H(P) \neq \varnothing$

Hence, a model characterised by $K1$–$K5$ is *consistent* with the preference model. The relation $\gtrsim$ is a partial preordering whenever it is derivable from $K(P)$. It is however not impossible, that $\gtrsim$ is not transitive for elements which are not preordered by $\bar{\mathsf{R}}$. There might exist elements x, y, z, such that

$x\bar{\not{\mathsf{R}}}y,\ y\bar{\not{\mathsf{R}}}x,\ x\bar{\not{\mathsf{R}}}z,\ z\bar{\not{\mathsf{R}}}x,\ y\bar{\not{\mathsf{R}}}z,\ z\bar{\not{\mathsf{R}}}y$
$x \gtrsim y,\ y \gtrsim z,\ z \gtrsim x.$

However, it is always *possible* that $\gtrsim$ is a preordering.

Remark 3.8.5

Note that most theorems of sections 3.4 and 3.5 also hold, since their proofs were based only on the revealed preference theorems.

Axioms $K6$ and $K7$ enable us to show that $\gtrsim$ is a complete preordering on X. First we show that $\bar{\mathsf{R}}^*$ and $\bar{\mathsf{R}}$ are complete on $\mathscr{P}$ and X.

Theorem 3.8.6

$\bar{\mathsf{R}}^*$ is a complete preordering.

Proof

Transitivity: holds by definition (property 3.5.14).
Completeness: by axiom $K6$.

Theorem 3.8.7

$\bar{\mathsf{R}}$ is a complete preordering.

Proof

Transitivity: holds by definition (property 3.4.14).
Completeness: By $K7$, for every x, y can be found P and Q such that $x \in K(P)$ and $y \in K(Q)$. By $K6$ $P\bar{\mathsf{R}}^*Q$ or $Q\bar{\mathsf{R}}^*P$, and this implies $x\bar{\mathsf{R}}y$ or $y\bar{\mathsf{R}}x$.

Theorem 3.8.8

$\gtrsim$ is a complete preordering.

Proof

By theorem 3.8.7, $\bar{\mathsf{R}}$ is a complete preordering. Hence we must prove

$x\bar{\mathsf{R}}y \Leftrightarrow x \gtrsim y$

By $K5$, $x\bar{\mathsf{R}}y \Rightarrow x \gtrsim y$. Suppose $x \not\bar{\mathsf{R}} y$. $\bar{\mathsf{R}}$ being complete, this means $y\bar{\mathsf{P}}x$, which implies by $K5$, $y \succ x$ and by definition

$y \succ x \Leftrightarrow x \not\gtrsim y \wedge y \gtrsim x.$

Remark 3.8.9

If $K7$ is dropped $\mathscr{P}$ remains completely preordered, but not X. But the set

$\{x | \exists P: x \in K(P)\}$

is completely preordered.

3.9 SOME MODIFICATIONS OF THE CHOICE FUNCTION MODEL

Axiom $K6$ of model K requires that $\mathscr{P}$ and $K(P)$ are such that $\bar{\mathsf{R}}^*$ is complete. If for $\mathscr{P}$ a more stringent condition holds, and if we use this condition as an axiom instead of $K6$, axiom $K4$ (strong axiom of revealed preference) can be dropped, because it then follows from the rest of the model.

The new axiom $K8$ requires that the union of every pair of two choice sets is again a choice set (see also theorem 3.5.16). This axiom is verified e.g. if $\mathscr{P}$ consists of all finite subsets of X.[8]

Axioms for model K^*

$K1$

$\forall P \in \mathscr{P}: K(P) \neq \varnothing$

$K2$

$\forall P \in \mathscr{P}: K(P) \subset P$

$K3$ (weak axiom of revealed preference)

$$\left.\begin{array}{ll} x \in K(P) & \wedge \\ y \in P & \wedge \\ y \notin K(P) & \end{array}\right\} \Rightarrow \nexists Q: y \in K(Q) \wedge x \in Q$$

$K5$ (transition axiom)

$x\bar{\mathsf{R}}y \Rightarrow x \gtrsim y$
$x\bar{\mathsf{P}}y \Rightarrow x \succ y$

$K7$

$\forall x \in X, \exists P \in \mathscr{P}: x \in K(P)$

$K8$

$\forall P, Q \in \mathscr{P}: (P \cup Q) \in \mathscr{P}$

8. This is the case treated in ARROW (1959).

It will now be shown that model K^* is sufficient for model K. $K1$, $K2$, $K3$, $K5$ and $K7$ being identical in both models, we only have to deduce $K4$ and $K6$.
We first give three auxiliary theorems.

Theorem 3.9.1

$$P \mathsf{I}^* Q \Leftrightarrow K(P) \cap Q = P \cap K(Q) \neq \varnothing$$

Proof

See the proof of theorem 3.5.6. This proof is only based on theorem 3.4.5, which is identical to $K3$.

Theorem 3.9.2

$$\forall P, Q \in \mathscr{P}: P \mathsf{I}^*(P \cup Q) \vee Q \mathsf{I}^*(P \cup Q)$$

Proof

Clearly, $(P \cup Q) \mathsf{R}^* P$ and $(P \cup Q) \mathsf{R}^* Q$.
Now of (a) and (b) at least one is true:
(a) $K(P \cup Q) \cap P \neq \varnothing$ which implies $P \mathsf{R}^*(P \cup Q)$,
(b) $K(P \cup Q) \cap P \neq \varnothing$ which implies $Q \mathsf{R}^*(P \cup Q)$.

Theorem 3.9.3

$$P \mathsf{R}^* Q \Rightarrow P \mathsf{I}^*(P \cup Q)$$

Proof

By theorem 3.9.2, $P \mathsf{I}^*(P \cup Q)$ or $Q \mathsf{I}^*(P \cup Q)$.
Suppose $Q \mathsf{I}^*(P \cup Q)$. Now by theorem 3.9.1,

$$K(Q) \cap (P \cup Q) = Q \cap K(P \cup Q) \neq \varnothing$$

We have

$$K(P \cup Q) \cap Q = K(Q) \cap (P \cup Q) \supset K(Q) \cap P \neq \varnothing$$

Hence

$K(P \cup Q) \cap Q \cap P \neq \varnothing$, hence $K(P \cup Q) \cap P \neq \varnothing$,

which means $P \mathsf{R}^*(P \cup Q)$.

By applying theorem 3.9.2 and using the transitivity of $\bar{\mathsf{R}}^*$ it follows

Theorem 3.9.4

$\forall P, Q \in \mathscr{P}: P\bar{\mathsf{R}}^*Q \vee Q\bar{\mathsf{R}}^*P.$

Proof

By theorem 3.9.2, at least one of (a) and (b) holds:

(a) $P\mathsf{I}^*(P \cup Q)$ and $(P \cup Q)\mathsf{R}^*Q$, hence $P\bar{\mathsf{R}}^*Q$.
(b) $Q\mathsf{I}^*(P \cup Q)$ and $(P \cup Q)\mathsf{R}^*P$, hence $Q\bar{\mathsf{R}}^*P$.

This proves axiom K6 for model K*
In the next theorem the strong axiom of revealed preference is deduced.

Theorem 3.9.5

$P\bar{\mathsf{R}}^*Q \Rightarrow Q\not{\mathsf{P}}^*P$

Proof

Suppose both relations $P\bar{\mathsf{R}}^*Q$ and $Q\mathsf{P}^*P$ hold. The first relation requires that $S_1, S_2, \ldots, S_n \in \mathscr{P}$ exist such that:

$P\mathsf{R}^*S_1 \wedge S_1\mathsf{R}^*S_2 \wedge \ldots \wedge S_n\mathsf{R}^*Q$

By theorem 3.9.3

$P\mathsf{I}^*(P \cup S_1)$

which implies (theorem 3.9.1)

$K(P) \cap (P \cup S_1) = K(P \cup S_1) \cap P$, or $K(P) \subset K(P \cup S_1)$.

Also

$(P \cup S_1)\mathsf{R}^*S_2 \qquad (\text{since } K(S_2) \cap S_1 \neq \varnothing)$

and by the same reasoning

$(P \cup S_1)\mathsf{I}^*(P \cup S_1 \cup S_2)$

and

$K(P \cup S_1) \subset K(P \cup S_1 \cup S_2)$

Hence

$K(P) \subset K(P \cup S_1 \cup S_2)$

Repeating this procedure we get

$K(P) \subset K(P \cup S_1 \cup S_2 \cup \ldots \cup S_n)$.

By hypothesis $K(P) \cap Q \neq \varnothing$, which implies

$K(P \cup S_1 \cup \ldots \cup S_n) \cap Q \neq \varnothing$

and since

$K(Q) \cap S_n \neq \varnothing$, $(P \cup S_1 \ldots \cup S_n)\mathrm{I}^*Q$

or

$K(P \cup S_1 \ldots \cup S_n) \cap Q = (P \cup S_1 \ldots \cup S_n) \cap K(Q)$

There must exist a point x such that

$x \in K(P) \cap Q \subset K(P \cup S_1 \cup S_2 \ldots \cup S_n) \cap Q$

or

$x \in (P \cup S_1 \ldots \cup S_n) \cap K(Q)$

or

$x \in P \cap K(Q)$

which contradicts $Q\mathrm{P}^*P$.

Thus we have proved:

model $K^* \Rightarrow$ model K

Finally, we show that in model K^* the weak axiom of revealed preference may be replaced by the axiom 'independence of irrelevant alternatives' (see 3.4.20):

$K9 \quad P, (P-T) \in \mathscr{P} \wedge T \subset X \wedge K(P) \not\subset T \Rightarrow$
$$K(P-T) = K(P) - T$$

Let model K^{**} consist of the axioms $K1$, $K2$, $K5$, $K7$, $K8$ and $K9$; then in the following theorem it is proved that

model $K^* \Leftrightarrow$ model K^{**}

Theorem 3.9.6

Given axioms $K1$, $K2$, $K5$, $K7$ and $K8$,

$K3 \Leftrightarrow K9$

Proof

$\Rightarrow$

Let $P, (P-T) \in \mathscr{P}$ and $K(P) \not\subset T$. Now $P\mathrm{I}^*(P-T)$, since $K(P-T) \subset P$ and $K(P) \cap (P-T) \neq \varnothing$. So $K(P-T) \cap P = (P-T) \cap K(P)$ and
$K(P-T) \cap P = K(P-T) = K(P) - T = K(P) \cap (P-T)$.

$\Leftarrow$

Suppose $x, y \in X$ and

$$\left.\begin{array}{l} x \in K(P) \wedge \\ y \in P \wedge \\ y \notin K(P) \end{array}\right\} \text{and} \begin{cases} x \in Q \\ y \in K(Q) \end{cases}$$

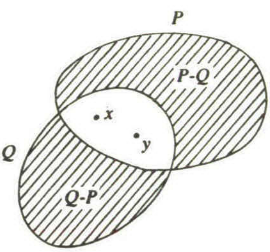


fig. 3.9.6

Now $K(P \cup Q) \neq \varnothing$ and

$$P = (P \cup Q) - (Q-P)$$
$$Q = (P \cup Q) - (P-Q)$$

Of (a) and (b) at least one must hold:

$$K(P \cup Q) \not\subset (Q-P) \qquad \text{(a)}$$
$$K(P \cup Q) \not\subset (P-Q) \qquad \text{(b)}$$

since a set can never be a part of two disjoint sets
(a) and (b) imply (a′) and (b′) respectively:

$$K(P) = K(P \cup Q) - (Q-P) \qquad \text{(a')}$$
$$K(Q) = K(P \cup Q) - (P-Q). \qquad \text{(b')}$$

Suppose (a′) holds, hence $x \in K(P)$ and $x \in K(P \cup Q)$. Then (b) holds too and therefore also (b′), hence $y \in K(Q)$.
It follows: $y \in K(P)$ and $x \in K(Q)$.

3.10 SUMMARY OF THE CONNECTIONS BETWEEN THE MODELS

In sections 3.4 and 3.5 was proved:

Model P (axioms $P1$, $P2$, $P3$, $P4$) implies axioms $K1$–$K5$ and $K9$.

In 3.8:

Axioms $K1$–$K5$ imply axioms $P1$ and $P2$

and

Model K (axioms $K1$–$K7$) implies model P.

In 3.9

Model K^* (axioms $K1$, $K2$, $K3$, $K5$, $K7$ and $K8$) implies model K.

and

Model K^{**} (axioms $K1$, $K2$, $K5$, $K7$, $K8$ and $K9$) is equivalent to model K^*.

Hence we also have

Model K^* implies model P

and

Model K^{**} implies model P.

4. Mathematics for consumer choice theory

4.1 INTRODUCTION

In the next two chapters will be presented the theory of consumer choice. In this theory, commodity bundles and prices are primitive concepts and both are represented by vectors in n-dimensional euclidean space. Therefore, some properties of this space are treated in the present chapter.

First, we treat some topological properties of sets in R^n. Some standard theorems will be left unproved, since the proofs can be found in any book on topology. In section 4.3, we introduce a special type of convex sets, namely c.u.p. sets, and with respect to these sets the concept of duality is defined and some properties of duals are deduced. Finally, we define continuity and concavity for real valued functions.

Many theorems are illustrated by figures in R^2, also if the theorem holds in R^n. These figures merely serve as illustrations, the proofs do not depend on them.

4.2 SETS IN REAL EUCLIDEAN SPACE

For every natural number n, the space R^n consists of all n-dimensional points $x = (x^1, x^2, \ldots, x^n)$ with real components x^i, $(i = 1, 2, \ldots, n)$. The points x are called n-vectors and R^n is a vector space, since vector addition and scalar multiplication are defined on this set:

$$x_1 \in R^n \wedge x_2 \in R^n \Rightarrow x_1 + x_2 \in R^n$$
$$\lambda \in R \wedge x \in R^n \Rightarrow \lambda x \in R^n$$

On this space is defined in the usual way the Euclidean distance $d(x, y)$

between two points x and y. This is a real valued function (see section 2.3) $d: R^n \times R^n \to R$.

Definition 4.2.1

If $x \in R^n$ and $y \in R^n$, the number $d(x, y)$ is said to be the (Euclidean) *distance* between x and y, if

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x^i - y^i)^2} = |x - y|$$

In R^2, $d(x, y)$ is the length of the line segment that connects x and y (fig. 4.2.2). The (necessary) properties of this distance are:

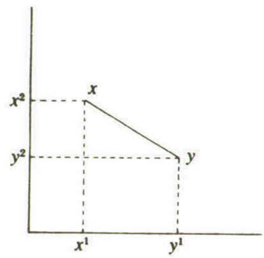


fig. 4.2.2

Theorem 4.2.3

$$\forall x, y: d(x, y) \geqslant 0 \qquad \text{(a)}$$
$$\forall x, y: d(x, y) = d(y, x) \qquad \text{(b)}$$
$$d(x, y) = 0 \Leftrightarrow x = y \qquad \text{(c)}$$
$$\forall x, y, z: d(x, y) + d(y, z) \geqslant d(x, z) \qquad \text{(d)}$$

We now consider some properties of sets that are subsets of some set X, which itself is a subset of R^n. (It is not excluded that $X = R^n$.)

Definition 4.2.4

If $x \in X \subset R^n$ and $\epsilon > 0$ is a positive real number, the set

$$B_\epsilon^X(x) = \{y \in X \mid d(x, y) < \epsilon\}$$

is said to be a (spherical) *neighbourhood* of the point x *with respect to* the set X.

Hence a spherical neighbourhood consists of the points of X, that lie within a certain distance from a given point. If $X = R^2$, $B_\epsilon^X(x)$ consists of the points within a circle with radius ϵ and centre x. If X is only a part of R^2, the neighbourhood only contains those points in the circle, that lie in X (see fig. 4.2.5).

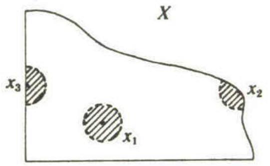


fig. 4.2.5

It should be noted that in general a neighbourhood is defined as any open set (see def. 4.2.8) containing a given point. In this study however, we only take into account spherical neighbourhoods.
If A is a subset of X and x is a point of A, then there may exist a neighbourhood of x that is entirely contained in A. Such a point is called an interior point of A (with respect to X). All interior points constitute the interior of A. In fig. 4.2.6 x is an interior point and y is not.

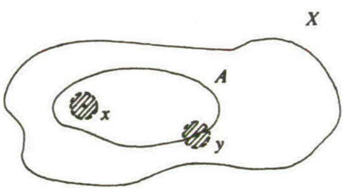


fig. 4.2.6

Definition 4.2.7

If $A \subset X \subset R^n$, the set

$$\mathrm{Int}_X A = \{x \in X \mid \exists\epsilon > 0: B_\epsilon^X(x) \subset A\}$$

is called the *interior* of A with respect to X; a point $x \in \mathrm{Int}_X A$ is called an *interior point* of A with respect to X.

Definition 4.2.8

A set $A \subset X \subset R^n$ is called an *open set* with respect to X, if

$$\mathrm{Int}_X A = A$$

Note that both the empty set $\varnothing$ and the set X itself are open with respect to X. Also $B_\epsilon^X(x)$ is open.

Definition 4.2.9

A set $A \subset X \subset R^n$ is called a *closed* set with respect to X, if the set $(X-A)$, the complement of A, is open.
Both $\varnothing$ and X are closed, hence they are simultaneously open and closed. If $X = R^n$, no other sets can be open and closed.[1] In this book we shall only handle subsets $X \subset R^n$, such that $\varnothing$ and X are the only sets that are simultaneously open and closed.

Theorem 4.2.10

a. The union of any collection of open sets is open.
b. The union of a finite collection of closed sets is closed.
c. The intersection of a finite collection of open sets is open.
d. The intersection of any collection of closed sets is closed.
The points that are neither in the interior of A, nor in the interior of $X-A$, are boundary points of A. Boundary points *may* be elements of A.

Definition 4.2.11

If $A \subset X \subset R^n$, the set

$$\mathrm{Bnd}_X A = X - (\mathrm{Int}_X A \cup \mathrm{Int}_X (X-A))$$

is called the *boundary* of A with respect to X; a point $x \in \mathrm{Bnd}_X A$ is called a *boundary point* of A.

Theorem 4.2.12

$\mathrm{Bnd}_X A = \mathrm{Bnd}_X (X-A)$ is closed. (a)
$\mathrm{Int}_X A \cap \mathrm{Bnd}_X A = \varnothing$. (b)
$A \cup \mathrm{Bnd}_X A = \mathrm{Int}_X A \cup \mathrm{Bnd}_X A$ is closed. (c)

Definition 4.2.13

If $A \subset X \subset R^n$, the set

$$\mathrm{Cl}_X A = A \cup \mathrm{Bnd}_X A$$

is called the *closure* of A with respect to X.

1. It is possible to choose X in such a way that all subsets of X are open and closed, e.g. if X is the set of vectors with only natural numbers as components.

Theorem 4.2.14

If $A \subset B \subset R^n$, we have

$$\mathrm{Bnd}_X A = \{x \in X \mid x \notin \mathrm{Int}_X A \wedge \forall \epsilon: B_\epsilon^X(x) \cap A \neq \varnothing\}$$
$$\mathrm{Cl}_X A = \{x \in X \mid \forall \epsilon: B_\epsilon^X(x) \cap A \neq \varnothing\}$$

Theorem 4.2.15

If $A \subset B \subset R^n$, we have

$$\mathrm{Int}_X A \subset \mathrm{Int}_X B$$
$$\mathrm{Cl}_X A \subset \mathrm{Cl}_X B.$$

Another important concept is the boundedness of a set

Definition 4.2.16

A set $A \subset R^n$ is called *bounded* if $x \in A$ and $\epsilon > 0$ exist, such that

$$A \subset B_\epsilon^{R^n}(x).$$

In the rest of this section we deal with convexity.[2] (All sets are in R^n; the index R^n for interiors, neighbourhoods etc. is omitted; ($[0, 1] = \{\lambda \in R \mid 0 \leqslant \lambda \leqslant 1\}$ denotes the unit interval.)

Definition 4.2.17

A set $C \subset R^n$ is called *convex*, if

$$x \in C \wedge y \in C \wedge \lambda \in [0, 1] \Rightarrow \lambda x + (1-\lambda) y \in C$$

This means that if two points are in C, all points on the line segment that connects the two points are also in C. Hence fig. 4.2.18a represents a convex set in R^2; b is not convex. Note that the space R^n is also convex. The intersection of any collection of convex sets is also convex.

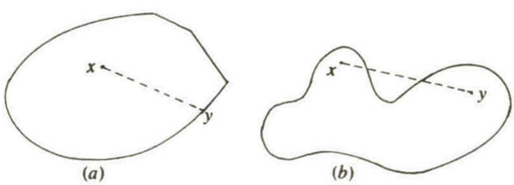


fig. 4.2.18

2. On convex sets, see e.g. BERGE, EGGLESTON, VALENTINE.

Theorem 4.2.19

If $C_i \subset R^n$ ($i \in I \subset N$) are convex, the intersection $\bigcap_{i \in I} C_i$ is convex.

Proof

Let $x \in \cap C_i$ and $y \in \cap C_i$.
If $z = \lambda x + (1-\lambda)y$ for $\lambda \in [0, 1]$, then $z \in C_i$, for every $i \in I$.
Hence $z \in \cap C_i$.

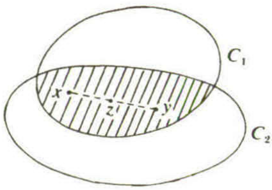


fig. 4.2.20

Theorem 4.2.21

If $C \subset R^n$ is convex, then Int C and Cl C are convex.

Proof

1. Let $x \in \text{Int } C$, $y \in \text{Int } C$ and $z = \lambda x + (1-\lambda)y$, for $\lambda \in [0, 1]$.
Then ϵ_1 and ϵ_2 exist, such that $B_{\epsilon_1}(x) \subset C$ and $B_{\epsilon_2}(y) \subset C$.
Let $\epsilon = \min(\epsilon_1, \epsilon_2)$, hence $B_\epsilon(x) \subset C$ and $B_\epsilon(y) \subset C$.
If $z_1 \in B_\epsilon(z)$, and if we choose $x_1 = x + (z_1 - z)$ and
$y_1 = y + (z_1 - z)$, then
$x_1 \in B_\epsilon(x)$, $y_1 \in B_\epsilon(y)$ and $z_1 = \lambda x_1 + (1-\lambda)y_1$.
So $z_1 \in C$, and therefore $B_\epsilon(z) \subset C$, and $z \in \text{Int } C$.

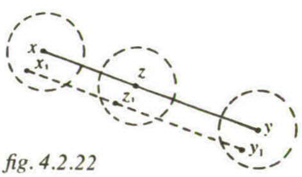


fig. 4.2.22

2. Let $x \in \text{Cl } C$ and $y \in \text{Cl } C$.
Suppose $z = \lambda x + (1-\lambda)y \notin \text{Cl } C$, for $\lambda \in [0, 1]$.
Then ϵ_0 exists, such that $B_{\epsilon_0}(z) \cap \text{Cl } C = \varnothing$, (since $B_{\epsilon_0}(z) \subset X - \text{Cl } C$).

Let

$t_1 \in B_{\epsilon_0}(x) \cap C \neq \emptyset$ and $t_2 \in B_{\epsilon_0}(x) \cap C$,

so that

$t = \lambda t_1 + (1-\lambda) t_2 \in C.$

But

$$|z-t| = |\lambda(x-t_1) + (1-\lambda)(y-t_2)| \\ \leqslant \lambda|x-t_1| + (1-\lambda)|y-t_2| \leqslant \epsilon_0.$$

Hence $t \in B_{\epsilon_0}(z)$ which is a contradiction.

Theorem 4.2.23

If C is a closed convex set, there cannot exist two closed sets A and B, such that

$$A \cup B = C \text{ and } A \cap B = \emptyset$$

By this theorem it is impossible to partition a closed convex set into two closed disjoint subsets. (In general sets with this property are called connected, hence the theorem implies that every convex set is connected.)

Definition 4.2.24

If $A \subset R^n$, the set

$$\operatorname{Conv} A = \{x \in R^n \mid \exists y, z \in A, \exists \alpha \in [0,1]: x = \alpha y + (1-\alpha) z\}$$

is called the *convex hull* of A.
Hence the convex hull of a set, is the set that is formed, if to the original set all points are added that lie on a line segment connecting two points of the set. It can be shown that Conv A is the smallest convex set that contains A. Obviously, the convex hull of a convex set is the set itself.

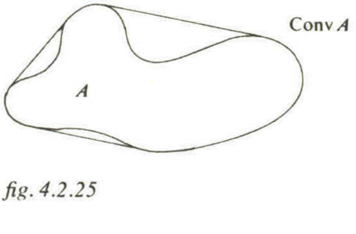


fig. 4.2.25

4.3 C.U.P. SETS

In the next two chapters consumer choice theory will be discussed. In this theory we only make use of non negative real vectors. Therefore, from now on X will denote the non negative part of Euclidean space:

$$X = \{x \in R^n \mid x \geqq 0\} = R_+{}^n \tag{4.3.1}$$

Now X is not a vector space, since on X scalar multiplication with negative real numbers is not defined. We have

$$x, y \in X \Rightarrow x+y \in X$$
$$\lambda \geqslant 0 \wedge x \in X \Rightarrow \lambda x \in X.$$

and

$$(x, y \in X \wedge x \geqq y) \Rightarrow x-y \in X$$

Since all neighbourhoods, open sets, etc. are defined with respect to X, the index X will be omitted.
Convex sets that are *unbounded above* and consist of *non negative* real vectors will be called c.u.p. sets. (*c*onvex, *u*nbounded, *p*ositive).

Definition 4.3.2

A set $C \subset R^n$ is called a *c.u.p. set*, if

C is convex (a)
$x \in C \wedge y \geqq x \Rightarrow y \in C$ (b)
$C \subset R_+{}^n$ (c)

In fig. 4.3.3 are drawn some c.u.p. sets in R^2.

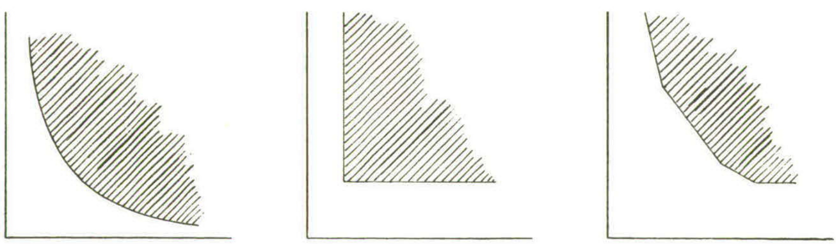


fig. 4.3.3

Theorem 4.3.4

If C_i $(i \in I \subset N)$ are c.u.p. sets, then also their intersection $\bigcap_{i \in I} C_i$ is a c.u.p. set.

Proof

Convexity holds by theorem 4.2.19.
If $x \in \cap C_i$ and $y \geqq x$, we have $y \in C_i$ for every $i \in I$, hence $y \in \cap C_i$.
Since all sets are subsets of X, the intersection is also contained in $X = R_+^n$.

Theorem 4.3.5

If C_1 and C_2 are c.u.p. sets, and $C_1 \cup C_2$ is convex, then $C_1 \cup C_2$ is a c.u.p. set.

Proof

We only have to prove unboundedness:
$y \geqq x \wedge x \in C_1 \cup C_2 \Rightarrow y \in C_1 \vee y \in C_2 \Rightarrow y \in C_1 \cup C_2$

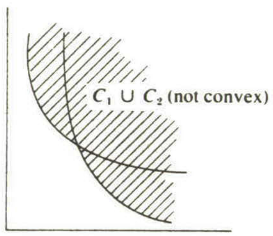


fig. 4.3.6

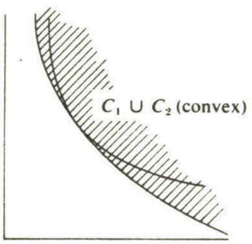


fig. 4.3.7

The following lemma is useful in many proofs.

Lemma 4.3.8

If $x \in X$ and $\lambda > 1$ and $x = \lambda y \in X$, then a neighbourhood $B_\epsilon(x)$ exists, such that

$$(z \in B_\epsilon(y) \Rightarrow z \geqq x) \wedge (z \in B_\epsilon(y) \wedge x > 0 \Rightarrow z > x)$$

Proof

Let $x_{\min} = \min\{x^i | x^i > 0 \wedge 1 \leqslant i \leqslant n\}$.
Choose $\epsilon < (1-\lambda)\, x_{\min}$, then for $x^i > 0$, we have

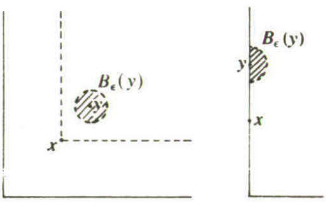


fig. 4.3.9

$-\epsilon < z^i - y^i < \epsilon$ and $y^i - x^i > \epsilon$,

hence

$z^i - x^i = (z^i - y^i) + (y^i - x^i) > 0$,

hence $z^i > x^i$
For $x^i = 0$, we have $z^i \geqslant x^i$.
Hence $z \geqq x$ and for $x > 0$ we have $z > x$.

Theorem 4.3.10

If C is a c.u.p. set, we have:

Int $C = \{x \in C \mid \exists \lambda: 0 < \lambda < 1 \wedge \lambda x \in C\}$ is a c.u.p. set	(a)
Cl C is a c.u.p. set	(b)
If C is closed, Bnd $C = \{x \in C \mid \lambda < 1 \Rightarrow \lambda x \notin C\}$	(c)
$x \in C \wedge C \neq X \Rightarrow \exists \lambda < 1: \lambda x \in$ Bnd C	(d)

Proof

(a) 1. Let $x \in \text{Int}\, C$, hence $\epsilon > 0$ exists, such that $B_\epsilon(x) \subset C$. If λ is chosen sufficiently close to 1 (e.g. $1 > \lambda > 1 - \epsilon/|x|$), we have $\lambda x \in B_\epsilon(x) \subset C$, hence

$$d(x, \lambda x) = |(1-\lambda)x| = (1-\lambda)|x| < (1-1+(\epsilon/|x|))|x| = \epsilon$$

2. Conversely, let $x \in C$ and $y = \lambda x \in C$ ($\lambda < 1$).
Hence $x = (1/\lambda)y$ and $(1/\lambda) > 1$. By lemma 4.3.8, $\epsilon > 0$ exists, such that

$$z \quad B_\epsilon(x) \Rightarrow z \geqq y$$

and since $y \in C$, we have $z \in C$. Hence $B_\epsilon(x) \subset C$ and x is an interior point. Int C is unbounded above, since

$$B_\epsilon(x) \subset \text{Int}\, C \wedge y \geqq x \Rightarrow B_\epsilon(y) \subset \text{Int}\, C$$

By theorem 4.2.21, Int C is convex.
(b) The closure of a convex set is closed.
If $x \in C$ and $y \geqq x$, we have $y \in C \subset \text{Cl}\, C$.
(c) Now $C = \text{Cl}\, C = \text{Int}\, C \cup \text{Bnd}\, C$ (theorem 4.2.12). Hence (c) follows by applying (a).
(d) Let $x \in C$ and $T = \{y \in X | \exists\lambda: 0 \leqslant \lambda \leqslant 1 \wedge y = \lambda x\}$.
T is closed and convex.
If $T_1 = T \cap \text{Cl}\, C$ and $T_2 = T \cap \text{Cl}\,(X - C)$, both T_1 and T_2 are closed and $T = T_1 \cup T_2$, $T_1 \neq \varnothing$ (since $x \in C$) and $T_2 \neq \varnothing$ (since $C \neq X$). Hence by theorem 4.2.23, $T_1 \cap T_2 \neq \varnothing$. Let $y \doteq \lambda x \in T_1 \cap T_2$, then $y \in \text{Bnd}\, C$.

Theorem 4.3.11

If $x, y, z \in \text{Bnd}\, C$, C is a c.u.p. set and $z = \alpha x + \beta y$ ($\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha + \beta \leqslant 1$ then $d(x, y) \geqslant d(x, z)$

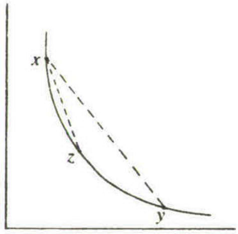


fig. 4.3.12

4.4 DUALITY[3]

In addition to $X = R_+{}^n$, we introduce a second set $P = R_+{}^n$. Though both coincide with $R_+{}^n$, they should be clearly distinguished. (In the next chapter X represents the commodity space and P the price space.) In some cases it will be useful to consider the points of X as column vectors and the points of P as row vectors.

On the cartesian product of P and X, we define a mapping into the non negative real numbers

$$f\colon\ P \times X \to R_+$$

Definition 4.4.1

If $p \in P$ and $x \in X$, we have

$$f(p, x) = px = \sum_{i=1}^{n} p^i x^i$$

For any fixed vector $p_0 \in P$ and any real number $\alpha > 0$, the set

$$\{x \in X \mid p_0 x = \alpha\} = \left\{x \in X \mid \left(\frac{1}{\alpha} p_0\right) x = 1\right\} \subset X$$

is an $n-1$-dimensional hyperplane in X. For all points of X above the plane we have $p_0 x > \alpha$, and for the points below the plane $p_0 x < \alpha$. If

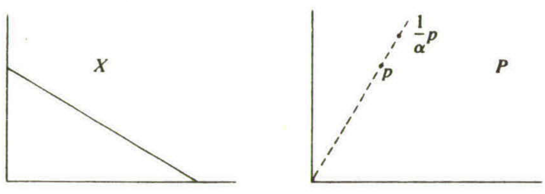


fig. 4.4.2

we always choose $\alpha = 1$, with any point of P can be associated a hyperplane. Thus we get a correspondence of P into X, denoted $L\colon P \to X$. In the same way is defined a correspondence $L\colon X \to P$.

Definition 4.4.3

If $p \in P$

$$L(p) = \{x \in X \mid px = 1\}.$$
$$M(p) = \{x \in X \mid px \leqslant 1\}.$$

3. See EGGLESTON, p. 25; VALENTINE, part V.

If $x \in X$

$$L(x) = \{p \in P \mid px = 1\}$$
$$M(x) = \{p \in P \mid px \leqslant 1\}$$

The sets $M(p)$ consist of all points on or below the hyperplane $L(p)$. The sets $L(p)$ and $M(p)$ have the following properties:

Theorem 4.4.4

$\forall p \in P$: $L(p)$ and $M(p)$ are closed and convex.
$L(p) = \text{Bnd}\, M(p)$.
$p > 0 \Rightarrow L(p)$ and $M(p)$ are bounded.
$L(0) = \varnothing$ and $M(0) = X$.

For the correspondences $L: P \to X$ and $M: P \to X$ can be stated:[4]

Theorem 4.4.5

$$p > q \Rightarrow L(p) \cap L(q) = \varnothing$$
$$p \geqq q \Rightarrow M(p) \subset M(q).$$

Clearly these properties also hold for $L(x)$ and $M(x)$.

Remark 4.4.6

The vector p considered as a vector in the space X is perpendicular to the plane $L(p)$. For if $px_0 = 0$, the vectors p and x_0 are perpendicular. The plane $L(p)$ is parallel to the plane $\{x \mid px = 0\}$.

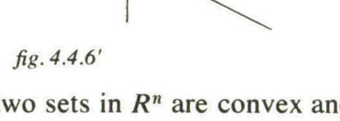


fig. 4.4.6'

If two sets in R^n are convex and do not intersect, there exists a hyperplane that *separates* them, i.e. one set lies above the plane and the other

4. It can also be shown that M and L are both lower semi-continuous correspondences and that both are upper semi-continuous for $p > 0$. For a definition of these properties, see e.g. BERGE, p. 114.

set lies below. This famous theorem is known as '*separation theorem*'. Without proof,[5] we present two different versions of the theorem. The first states that any two convex sets that do not intersect, can be separated by a plane that does not contain interior points of the sets, whereas by the second, two closed convex sets, of which one is also bounded, can be separated by a plane such that the plane does not contain any point of one of the sets.

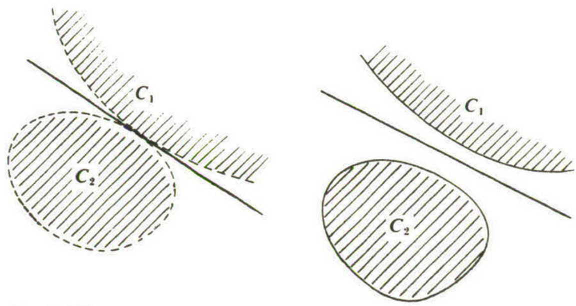


fig. 4.4.7

Theorem 4.4.8

a. If $C_1 \subset R^n$ and $C_2 \subset R^n$ are convex and $C_1 \cap C_2 = \emptyset$, $p \in R^n$ and $\alpha \in R$ exist, such that

$$x \in C_1 \Rightarrow px \geqslant \alpha \text{ and } x \in C_2 \Rightarrow px \leqslant \alpha$$

b. If also C_2 is closed and bounded and C_1 is closed, $p \in R^n$ and $\alpha \in R$ exist, such that

$$x \in C_1 \Rightarrow px > \alpha \text{ and } x \in C_2 \Rightarrow px < \alpha$$

If $\alpha \neq 0$, the separating hyperplane can be written $L((1/\alpha)p)$. $\alpha > 0$ always holds if $0 \in X$ is an interior point of one of both sets, since the plane $\{x \in R^n | px = 0\}$ passes through the origin.

Obviously, the theorem also holds if one of both sets is only a single point. Hence, any point that is not an element of a convex set can be separated from that set by a hyperplane. From this it directly follows, that a plane passes through such a point that does not intersect the set: Let $C_2 = \{y\}$ and $py = \beta \leqslant 1$, while $x \in C_1 \Rightarrow px \geqslant 1$.

Now

$$\frac{1}{\beta} py = 1 \text{ and } x \in C_1 \Rightarrow \frac{1}{\beta} px \geqslant px \geqslant 1$$

5. See e.g. BERGE, p. 171, BERGE and GHOUILA-HOURI, p. 52, EGGLESTON, p. 19, VALENTINE, p. 27.

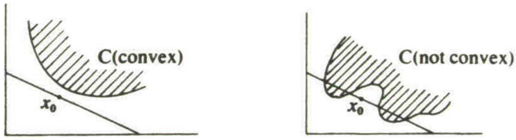


fig. 4.4.9

If C is a *closed* c.u.p. set and $p \in P$, three cases can occur:
1. $L(p)$ contains interior points of C:

$$L(p) \cap \operatorname{Int} C \subset M(p) \cap \operatorname{Int} C \neq \varnothing$$

(or equivalently, $\exists x\colon x \in \operatorname{Int} C \wedge px = 1$)
2. $L(p)$ does not contain interior points, but it does contain boundary points of C:

$$L(p) \cap \operatorname{Int} C = M(p) \cap \operatorname{Int} C = \varnothing$$
$$L(p) \cap \operatorname{Bnd} C = M(p) \cap \operatorname{Bnd} C \neq \varnothing$$

(or equivalently, $x \in \operatorname{Int} C \Rightarrow px > 1 \wedge \exists x\colon x \in \operatorname{Bnd} C \wedge px = 1$)
3. $L(p)$ does not contain any point of C:

$$L(p) \cap C = M(p) \cap C = \varnothing. \quad (\text{or}, x \in C \Rightarrow px > 1)$$

In the second case $L(p)$ is called a support of C, and if x is a point of the intersection, $L(p)$ is said to support C in x.

Definition 4.4.10

If C is a closed c.u.p. set and if

$$L(p) \cap \operatorname{Int} C = \varnothing \wedge L(p) \cap C \neq \varnothing$$

then $L(p)$ is said to be a *support* of C.
In the third case, it is possible that, though $L(p)$ does not support C, $L(p)$ 'touches' C in the infinite; in this case we call $L(p)$ an asymptote of C.

Definition 4.4.11

If C is a closed c.u.p. set and $p \in P$, and if

$$L(p) \cap C = \varnothing \wedge (q \geqq p \Rightarrow L(q) \cap \operatorname{Int} C \neq \varnothing)$$

then $L(p)$ is said to be an *asymptotic support* or an asymptote of C.
With every closed c.u.p. set C in X can be associated a dual set in P that contains all $p \in P$ for which $L(p)$ does not contain interior points of C.

Definition 4.4.12

If $C \subset X$ is a closed c.u.p. set, the set

$$C^* = \{p \in P | L(p) \cap \operatorname{Int} C = \emptyset\}$$

is called the *dual* of C in P.
(By interchanging X and P, respectively x and p, we get the dual of a set in P, which is a set in X.)

Theorem 4.4.13

If $0 \notin C$ and C is a closed c.u.p. set, we have $C^* \neq \emptyset$.

Proof

Since $0 \notin C$, by theorem 4.4.8 a hyperplane $L(p)$ that separates 0 and C exists, such that $L(p) \cap C = \emptyset$; hence $p \in C^*$.

The set X has the empty set as its dual (since $0 \in X$) and the empty set $\emptyset \subset X$ has dual P.

Theorem 4.4.14

The dual C^* of a closed c.u.p. set C is also a closed c.u.p. set.

Proof

1. C^* is convex:
Let $p_1 \in C^*$ and $p_2 \in C^*$, hence

$$x \in C \Rightarrow p_1 x \geqslant 1 \wedge p_2 x \geqslant 1$$

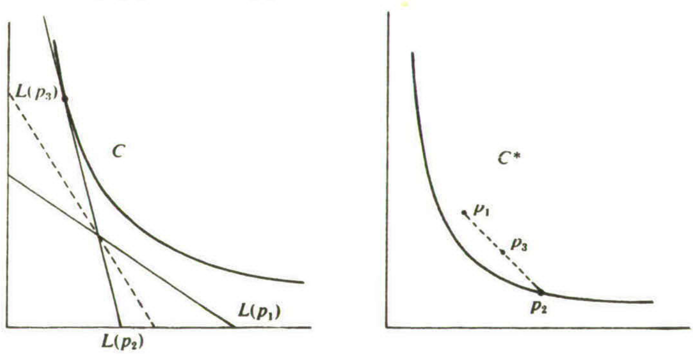


fig. 4.4.15

For $\lambda \in [0, 1]$, it follows

$$x \in C \Rightarrow (\lambda p_1 + (1-\lambda)p_2)x = \lambda p_1 x + (1-\lambda)p_2 x \geqslant 1.$$

2. C^* is unbounded from above:

Let $p \in C^*$, hence, $x \in C \Rightarrow px \geqslant 1$ and therefore $q \geqq p \Rightarrow qx \geqslant px \geqslant 1$, hence $q \in C^*$.

3. Since $C^* \subset P$, points of C are non negative.

4. C^* is closed: we show that $P - C^*$ is open.
Let $p \in P - C^*$. Now $x \in \operatorname{Int} C \cap L(p)$ exists, hence $\lambda < 1$ exists, such that $\lambda x \in \operatorname{Int} C$ (theorem 4.3.10).
Choose

$$\epsilon = \frac{1-\lambda}{\lambda |x|},$$

then

$$q \in B_\epsilon(p) \Rightarrow q\lambda x < 1$$

since

$$q\lambda x = \lambda(q-p)x + \lambda px = \lambda(q-p)x + \lambda \leqslant \lambda|q-p||x| + \lambda$$
$$< \lambda\epsilon|x| + \lambda = \lambda \frac{1-\lambda}{\lambda|x|}|x| + \lambda = 1.$$

Hence $p \in \operatorname{Int} P - C^*$ and $P - C^*$ is open.

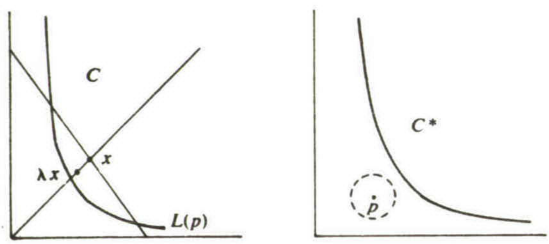


fig. 4.4.16

Remark 4.4.17

The set $\{p | L(p) \cap \operatorname{Int} C = \varnothing\}$ is also a closed c.u.p. set, if C is any subset of X that is unbounded above. It needs not to be convex and closed.

Theorem 4.4.18

If C^* is the dual of a closed c.u.p. set, $p \in \text{Bnd } C^*$ and $p > 0$, then $L(p)$ is a support of C.

Proof

Let $p \in \text{Bnd } C^*$. Choose $\lambda < 1$. Since $\lambda p \notin C^*$, we have $L(\lambda p) \cap \text{Int } C \neq \varnothing$. The set $K = M(\lambda p) \cap C$ is closed and bounded. Choose x_0 such that $px_0 = \min\limits_{x \in K} px$. This minimum exists, since px is continuous (theorem 4.6.2 below). We must have $px_0 = 1$. Suppose $px_0 = \alpha > 1$; then $(1/\alpha)p_0 \in C^*$, and this is a contradiction.

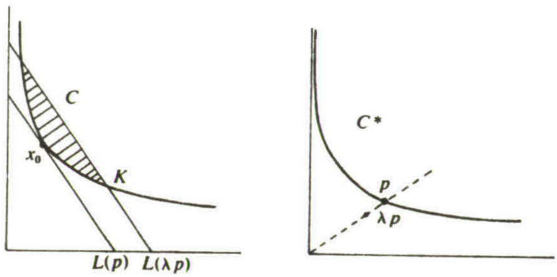


fig. 4.4.19

Boundary points of C^* that have 0-components, need not support C, because they *may* be asymptotes of C (see fig. 4.4.20).

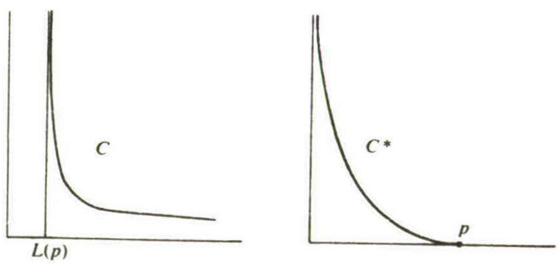


fig. 4.4.20

The last theorem directly implies:

Theorem 4.4.21

If C is a closed c.u.p. set

$$\text{Bnd } C^* \supset \{p \in P \mid L(p) \cap C \neq \varnothing \wedge L(p) \cap \text{Int } C = \varnothing\}.$$

$$\text{Int}\, C^* \subset \{p \in P \mid L(p) \cap C = \varnothing\}.$$

C and C^* are sets of the same type. For $C^* \subset P$, a dual in X is defined (see definition 4.4.11):

$$C^{**} = \{x \in X \mid L(x) \cap \text{Int}\, C^* = \varnothing\}.$$

It will be shown that the set C^{**} is identical to the original set C.

Theorem 4.4.22

If $C \subset X$ is a closed c.u.p. set, $C^* \subset P$ is its dual and $C^{**} \subset X$ is the dual of C^*, then

$$C = C^{**}$$

Proof

1. Let $x \in C$.
We have $p \in L(x) \Rightarrow x \in L(p)$, hence $p \in L(x) \Rightarrow x \in L(p) \cap C \neq \varnothing$, and this implies by theorem 4.4.21, $p \notin \text{Int}\, C^*$

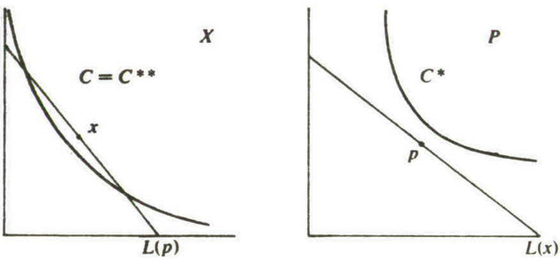


fig. 4.4.23

It follows

$$p \in L(x) \Rightarrow L(x) \cap \text{Int}\, C^* = \varnothing,$$

hence $x \in C^{**}$
2. Let $x \notin C$. By theorem 4.4.8, $p \in P$ exists, such that

$y \in C \Rightarrow py > 1$ and $px = \lambda < 1$; hence $L(p) \cap C = \varnothing$, and so $p \in C^*$.

We also have $(1/\lambda)px = 1$ and $L((1/\lambda)p) \cap C = \varnothing$, hence $(1/\lambda)p \in C^*$, and so by theorem 4.3.10, $(1/\lambda)p \in \text{Int}\, C^*$.
This implies $(1/\lambda)p \in L(x) \cap C^* \neq \varnothing$, hence $x \notin C^{**}$.

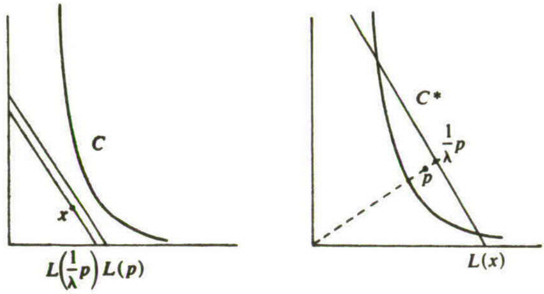


fig. 4.4.24

This theorem directly implies:

Theorem 4.4.25

$L(p)$ supports C in $x \Leftrightarrow L(x)$ supports C^* in p.

Proof

If $L(p)$ supports C in x, then $p \in \text{Bnd}\, C^*$.
Since $x \in C^{**} = C$, by definition, $L(x) \cap \text{Int}\, C^* = \emptyset$; since $px = 1, L(x) \cap C^* \neq \emptyset$.

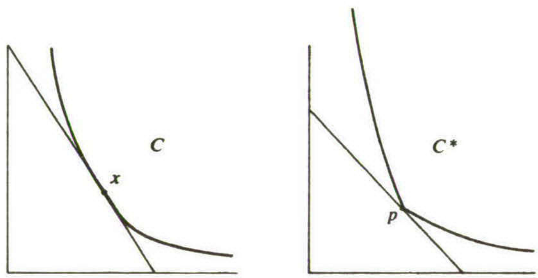


fig. 4.4.26

The converse is proved in the same way.

Remark 4.4.27

If $x \in C$ is supported by different supporting hyperplanes $L(p)$ and $L(q)$, then p and q are in the intersection $L(x) \cap \text{Bnd}\, C^*$, and this intersection is convex. If $L(p)$ supports C in different points $y \in C$, then all $L(y)$ support C^* in p.
If in R^2, Bnd C consists of line segments, then also C^*. Now the

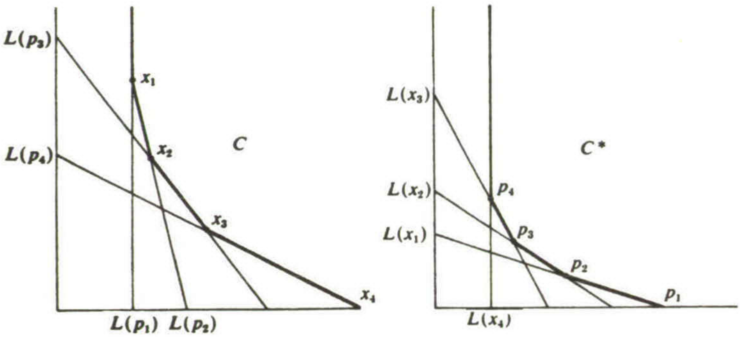


fig. 4.4.28

vertices of Bnd C correspond to line segments of Bnd C^* and vice versa.

By definition 4.4.3

$$M(x) = \{p \mid px \leqslant 1\}.$$

and (see 2.3.3)

$$M(C) = \bigcup_{x \in C} M(x) = \{p \mid \exists x\colon p \in M(x)\}.$$

Theorem 4.4.29

If C is a c.u.p. set in X

$$M(\operatorname{Int} C) = \operatorname{Int} M(C)$$

Proof

1. $p \in M(\operatorname{Int} C)$, hence $x \in \operatorname{Int} C$ exists, such that $px \leqslant 1$. Hence by theorem 4.3.10, $\lambda < 1$ exists such that $\lambda x \in C$. Choose

$$\eta < \frac{1-\lambda}{\lambda|x|}.$$

Then for $q \in B_\eta(p)$

$$q\lambda x = \lambda(p+q-p)x = \lambda(p+(q-p))x \leqslant \lambda(1+|q-p|\,|x|) <$$

$$\lambda\left(1+\frac{1-\lambda}{\lambda|x|}|x|\right) = 1$$

This implies $B_\eta(p) \subset M(C)$, hence $p \in \operatorname{Int} M(C)$.
2. Let $p \in \operatorname{Int}\ M(C)$, hence $\epsilon > 0$ exists, such that $B_\epsilon(p) \subset$

$M(C)$. Choose $\lambda > 1$, such that $\lambda p \in B_\epsilon(p)$. Let $y \in M(\lambda p) \cap C$ and since C is a c.u.p. set $(1/\lambda)y \in M(p) \cap \operatorname{Int} C$, hence $p \in M(\operatorname{Int} C)$.

This implies

Theorem 4.4.30

If C^* is the dual of a closed c.u.p. set C

$$C^* = P - \operatorname{Int} M(C)$$

Proof

$$\begin{aligned} M(\operatorname{Int} C) &= \{p \mid \exists x \in \operatorname{Int} C: p \in M(x)\} \\ &= \{p \mid \exists x \in \operatorname{Int} C: x \in M(p)\} \\ &= \{p \mid M(p) \cap \operatorname{Int} C \neq \varnothing\} = P - C^*. \end{aligned}$$

Theorem 4.4.31

If C_1 and C_2 are closed c.u.p. sets,

$$C_1 \subset C_2 \Rightarrow C_2^* \subset C_1^*$$

Proof

Let $p \in C_2^*$, hence $L(p) \cap \operatorname{Int} C_2 = \varnothing$.
Since $C_1 \subset C_2$, $\operatorname{Int} C_1 \subset \operatorname{Int} C_2$, and we have $L(p) \cap \operatorname{Int} C_1 = \varnothing$, hence $p \in C_1^*$.

Theorem 4.4.32

If C_1 and C_2 are closed c.u.p. sets and $C_1 \cup C_2$ is convex

$$(C_1 \cup C_2)^* = C_1^* \cap C_2^* \qquad \text{(a)}$$
$$(C_1 \cap C_2)^* = C_1^* \cup C_2^* \qquad \text{(b)}$$

Proof

(a) 1. $C_1 \subset C_1 \cup C_2$, hence $C_1^* \supset (C_1 \cup C_2)^*$, (theorem 4.4.31)
$C_2 \subset C_1 \cup C_2$, hence $C_2^* \supset (C_1 \cup C_2)^*$

and this implies

$$C_1^* \cap C_2^* \supset (C_1 \cup C_2)^*$$

2. Let $p \in C_1^* \cap C_2^*$, then we have

$x \in C_1 \Rightarrow px \geqslant 1$ and $x \in C_2 \Rightarrow px \geqslant 1$

it follows

$x \in (C_1 \cup C_2) \Rightarrow px \geqslant 1$, hence $p \in (C_1 \cup C_2)^*$

and this implies

$C_1^* \cap C_2^* \subset (C_1 \cup C_2)^*$.

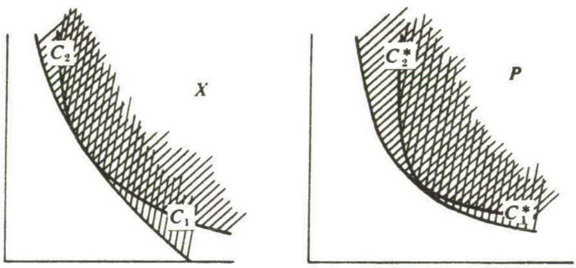


fig. 4.4.33

(b) By theorem 4.4.22, we have

$(C_1 \cap C_2)^* = (C_1^{**} \cap C_2^{**})^*$

$(C_1^* \cup C_2^*) = (C_1^* \cup C_2^*)^{**}$

By (a)

$((C_1^* \cup C_2^*)^*)^* = (C_1^{**} \cap C_2^{**})^*$

hence

$(C_1 \cap C_2)^* = (C_1^* \cup C_2^*)$.

Remark 4.4.34

The first part of the theorem also holds if the union is not convex (see remark 4.4.17).

4.5 A THEOREM ON C.U.P. SETS

In this section a theorem will be presented, that we need in the last chapter to prove the completeness of the revealed preference relation. We can only prove this theorem directly in R^2. Therefore, we first show how c.u.p. sets in R^n can be reduced to sets in R^2.

Let $s_1 \in R^n$ and $s_2 \in R^n$ be two row vectors, that establish a $2 \times n$-matrix $(n \geqslant 2)$.

$$S = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} s_1{}^1, s_1{}^2, \ldots, s_1{}^n \\ s_2{}^1, s_2{}^2, \ldots, s_2{}^n \end{bmatrix} \tag{4.5.1}$$

The vectors s_i satisfy

$$\begin{aligned} & s_1 \neq s_2 \\ & s_1 \geqq 0, s_1 \not> 0, s_2 \geqq 0, s_2 \not> 0 \\ & s_1 + s_2 > 0 \end{aligned} \tag{4.5.2}$$

Hence they are non negative and each has at least one 0-component but not the same.
With the help of S the space $X = R^n$ is mapped[6] into R^2, while a set in R^2 exists that corresponds to points of P:

$$\begin{aligned} & \hat{X} = \{\hat{x} | \exists x \in X: \hat{x} = Sx\} \\ & \hat{P} = \{\hat{p} | \exists p \in P: p = \hat{p}S\} \end{aligned} \tag{4.5.3}$$

Theorem 4.5.4

$\hat{X} = R_+{}^2$ and $\hat{P} = R_+{}^2$,

Proof

1. Let $x \in X$. Since $x \geqq 0$ and $S \geqq 0$, we have $\hat{x} = Sx \geqq 0$.
If conversely $\hat{x} \in R^2$ and $\hat{x} \geqq 0$, $x \in R_+{}^n$ exists such that $\hat{x} = Sx$.
2. If $\hat{p} \in R_+{}^2$, $\hat{p}S \geqq 0$.
If $\hat{p} \ngeqq 0$, we have $\hat{p}^1 < 0$ or $\hat{p}^2 < 0$; for $p = \hat{p}S$ $p^i < 0$ $(i = 1, 2, \ldots, n)$ exists, by condition (4.5.2).

Let $C \subset X$ be a closed c.u.p. set. By the matrix S this set is mapped into $\hat{X}$:

$$\hat{C} = \{\hat{x} \in \hat{X} | \exists x \in C: \hat{x} = Sx\} \tag{4.5.5}$$

Theorem 4.5.6

The set $\hat{C} \subset \hat{X}$ is a closed c.u.p. set.

6. Remember that vectors in X are considered as column vectors and vectors in P as row vectors. Hence $\hat{p}S$ is the product of a row vector and a matrix and Sx is the product of a matrix and a column vector.

Proof

1. $\hat{C}$ is convex:
Let $\hat{x}, \hat{y} \in \hat{C}$, now $x, y \in C$ exist such that $\hat{x} = Sx$ and $\hat{y} = Sy$. Since $x + (1-\lambda)y \in C$ we have $S(\lambda x + (1-\lambda)y) = \lambda Sx + (1-\lambda)Sy = \lambda\hat{x} + (1-\lambda)\hat{y} \in \hat{C}$
2. $\hat{C}$ is unbounded:
Let $\hat{x} = Sx \in \hat{C}$ and $\hat{y} \geqq \hat{x}$, hence $\hat{v} = \hat{y} - \hat{x} \geqq 0$ and $\hat{v} \in \hat{X}$.
Now by theorem 4.5.4, $v \in X$ exists such that $\hat{v} = Sv$.
If $y = x + v$, then $y \in C$, hence $Sy = Sx + Sv = \hat{y} \in \hat{C}$.
3. The points of $\hat{C}$ are non negative, since $\hat{C} \subset \hat{X}$.
4. $\hat{C}$ is closed:
Let $\hat{x} \in \mathrm{Bnd}\, \hat{C}$ and suppose $\hat{x} \notin \hat{C}$.
Clearly, the set $V = \{x \in X | \hat{x} = Sx\}$ is closed and convex.
We first show that V is bounded:
For $x \in V$ we have simultaneously
$\hat{x}^1 = s_1 x$ and $\hat{x}^2 = s_2 x$, $((\hat{x}^1, \hat{x}^2) = \hat{x})$.
Hence $\hat{x}^1 + \hat{x}^2 = (s_1 + s_2)x$ and
$V \subset \{x \in X |\ (s_1 + s_2)x = \hat{x}^1 + \hat{x}^2\}$

Since $(s_1 + s_2) > 0$, this last set is bounded and therefore also V.
By the separation theorem, the closed and bounded convex set V and the closed convex set C are *strictly* separated by a hyperplane. Let $L(p)$ be this plane and $p = \hat{p}S$, then

$x \in V \Rightarrow px < 1 \Leftrightarrow \hat{p}Sx < 1$
$y \in C \Rightarrow py > 1 \Leftrightarrow \hat{p}Sy > 1$

Since $Sx = \hat{x}$ for every $x \in V$, we have $\hat{p}\hat{x} < 1$. Taking into account the definition of $\hat{C}$,

$\hat{y} \in C \Rightarrow \hat{p}\hat{y} > 1$

Hence the hyperplane $L(\hat{p})$ *strictly* separates $\hat{x}$ from $\hat{C}$, but this is impossible since $\hat{x} \in \mathrm{Bnd}\, \hat{C}$.
Hence $V \cap C \neq \emptyset$ and $\hat{x} \in \hat{C}$.

The set $\hat{C} \subset \hat{X}$ has a dual $\hat{C}^* \subset \hat{P}$ where

$$\hat{C}^* = \{\hat{p} \in \hat{P} | L(\hat{p}) \cap \mathrm{Int}\, \hat{C} = \emptyset\}$$

On the other hand, a set $\hat{C}^0$ in $\hat{P}$ exists such that its points are mapped into C^* by the matrix S:

$$\hat{C}^0 = \{\hat{p} \in \hat{P} | \exists p \in C^* : p = \hat{p}S\} \tag{4.5.7}$$

Theorem[7] 4.5.8

If C is a closed c.u.p. set in X

$$\hat{C}^0 = \hat{C}^*$$

Proof

1. $\hat{C}^0 \subset \hat{C}^*$.
Let $\hat{q}_0 \in \hat{C}^0$, hence $q_0 = \hat{q}_0 S \in C^*$.
For every $\hat{y} \in \hat{C}$, a point $y \in C$ exists, such that $\hat{y} = Sy$, and for y holds $q_0 y \geqslant 1$. This implies $q_0 y = \hat{q}_0 S y = \hat{q}_0 \hat{y} \geqslant 1$, hence $\hat{q}_0 \in \hat{C}^*$.
2. $\hat{C}^* \subset \hat{C}^0$.
Let $\hat{q}_0 \in \hat{C}^*$ and $q_0 = \hat{q}_0 S$
Suppose $\hat{q}_0 \notin \hat{C}^0$, hence $q_0 \notin C^*$
A point $z \in C$ exists, such that $q_0 z < 1$.
But if $\hat{z} = Sz$, we have $\hat{z} \in \hat{C}$ and $\hat{q}_0 \hat{z} = q_0 S \hat{z} = q_0 z < 1$,
hence $\hat{q}_0 \notin \hat{C}^*$
and this is a contradiction.

This theorem directly implies:

$$q \in \text{Bnd}\, C^* \wedge q = \hat{q}S \Rightarrow \hat{q} \in \text{Bnd}\, \hat{C}^*$$

In the following theorem, it is assumed that a c.u.p. set is a subset of a second c.u.p. set and that these sets have at least one common boundary point, but do not coincide.

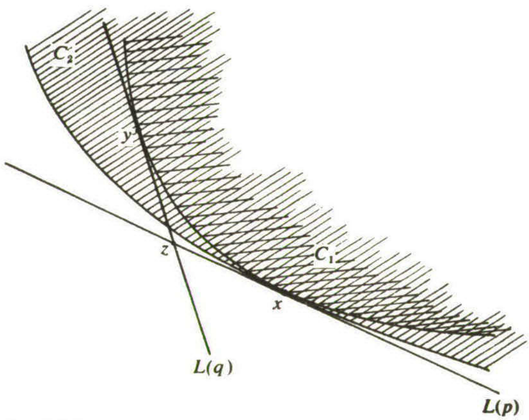


fig. 4.5.9

7. This theorem on c.u.p. sets is similar to a result in MALINVAUD (1964), with respect to ellipsoids, see p. 154.

Theorem 4.5.10

If C_1 and C_2 are two closed c.u.p. sets in X, such that

$C_1 \subset C_2$ (a)
$x \in \text{Bnd } C_1 \cap \text{Bnd } C_2$ and $L(p)$ supports C_2 in x (b)
$y \in \text{Bnd } C_1$ and $y \notin \text{Bnd } C_2$ and $L(q)$ supports C_1 in y $(q > 0)$ (c)

then $L(r_1)$ exists which supports C_1 in t_1, such that if $L(\tau r_2)$ supports C_2,

$$t_2 \in L(\tau r_1) \cap \text{Bnd } C_2 \Rightarrow pt_2 \geqslant pt_1$$
$$r_1 = \alpha p + \beta q \text{ for } \alpha > 0, \beta > 0 \text{ and } \alpha + \beta < 1$$

Proof

This theorem is first proved for $X = R^2$.
A point z exists, such that

$$pz = qz = 1 = px = qy$$

Now any point $t \in X$ can be written

$$t = z + \lambda(y-z) + \mu(x-z)$$

We introduce two real valued functions μ_1: $[0, 1] \to R$ and μ_2: $[0, 1] \to R$. These functions express μ_1 and μ_2 in terms of λ:

$$\mu_1(\lambda) = \min\{\mu | t = z + \lambda(y-z) + \mu(x-z) \wedge t \in C_1\}$$
$$\mu_2(\lambda) = \min\{\mu | t = z + \lambda(y-z) + \mu(x-z) \wedge t \in C_2\}$$

Since C_1 and C_2 are closed and convex, these functions are continuous.
With the help of these functions we construct two mappings of $[0, 1]$ into X:

$$t_1(\lambda) = z + \lambda(y-z) + \mu_1(\lambda)(x-z)$$
$$t_2(\lambda) = z + \lambda(y-z) + \mu_2(\lambda)(x-z)$$
$$= t_1(\lambda) - [\mu_1(\lambda) - \mu_2(\lambda)](x-z)$$

Obviously, $t_1(\lambda)$ and $t_2(\lambda)$ are boundary points of C_1 and C_2. We have

$$pt_1(\lambda) = pt_2(\lambda) = pz + \lambda p(y-z) > 1$$

and

$$0 = \mu_1(1) > \mu_2(1) \text{ since } z + (y-z) = y \in \text{Int } C_2$$
$$\mu_1(\lambda) \geqslant \mu_2(\lambda) \text{ for } 0 \leqslant \lambda \leqslant 1, \text{ since } C_1 \subset C_2$$

$$\mu_1(0) = \mu_2(0) \text{ since } z + (x - z) = x \in \text{Bnd } C_1 \cap \text{Bnd } C_2$$

If we define

$$\mu(\lambda) = \mu_1(\lambda) - \mu_2(\lambda)$$

then this function is nowhere negative, not everywhere 0 and it is 0 for $\lambda = 0$. Further, it is continuous on the closed and bounded set $[0, 1]$, and hence it achieves a maximum in this interval:

$$\mu(\lambda') = \max_{\lambda \in [0,1]} \mu(\lambda) > 0$$

Now we choose λ^0 such that

$$\lambda^0 = \min \{\lambda' \mid \mu(\lambda') = \max_{\lambda \in [0,1]} \mu(\lambda)\}$$

Let $L(r_1)$ support C_1 in

$$t_1(\lambda^0) = z + \lambda^0(z - x) + \mu(\lambda^0)(x - z)$$

Hence a support $L(\tau r_1)$ of C_2 in a point v exists, and we have $\tau > 1$, since $C_1 \subset C_2$.
We most prove that $pv \geqslant pt_1(\lambda^0)$.
Suppose

$$pv < pt_1(\lambda^0) = pt_2(\lambda^0)$$

then for $v = t_2(\lambda)$:

$$pv = pz + \lambda p(y - z) < pz + \lambda^0 p(y - z)$$

Hence $\lambda < \lambda^0$.
Since $\mu(\lambda^0) > \mu(\lambda)$

$$r_1 t_1(\lambda^0) = 1 \leqslant r_1 t_1(\lambda)$$

and this implies

$$\begin{aligned} 1 &= \tau r_1 t_2(\lambda) = \tau r_1[t_1(\lambda) - \mu(\lambda)(x - z)] \\ &= \tau r_1 t_1(\lambda) - \tau\mu(\lambda) r_1(x - z) > \tau r_1 t_1(\lambda^0) - \tau\mu(\lambda^0) r_1(x - z) \\ &= \tau r_1[t_1(\lambda^0) - \mu_1(\lambda^0)(x - z)] = \tau r_1 t_2(\lambda^0) \end{aligned}$$

Hence $\tau r_1 t_2(\lambda^0) < 1$, and therefore $L(\tau r_1) \cap \text{Int } C_2 \neq \varnothing$, but this is impossible, since $L(\tau r_1)$ is a support of C_2.
Hence we have

$$v \in L(\tau r_1) \cap \text{Bnd } C_2 \Rightarrow pv \geqslant 1$$

It remains to prove that α and β exist, such that

$$r_1 = \alpha p + \beta q \quad (\alpha > 0, \beta > 0 \text{ and } \alpha + \beta \leqslant 1)$$

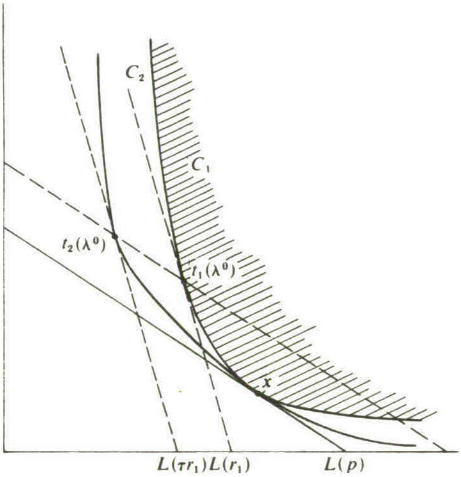


fig. 4.5.11

Let $pt_1(\lambda^0) = \dfrac{1}{\epsilon} > 1$ and $qt_1(\lambda^0) = \dfrac{1}{\eta} > 1$,

hence

$$\epsilon px < \epsilon pt_1(\lambda^0) = 1 \leqslant \epsilon py$$
$$\eta qx > \eta qt_1(\lambda^0) = 1 \geqslant \eta qy$$

(since $pt_1(\lambda^0) = pz + \lambda^0 p(y-z) \leqslant pz + p(y-z)$).
Since $\epsilon pt_1(\lambda^0) = \eta qt_1(\lambda^0) = r_1 t_1 = 1$ and $n = 2$

$$r_1 = \gamma(\epsilon p) + \delta(\eta q) \quad \text{for} \quad \gamma + \delta = 1.$$

Suppose $\alpha = \gamma\eta < 0$, hence $\gamma < 0$.
Since $\gamma\epsilon py < \gamma$ and $\delta\eta qy < \delta$, this implies

$$r_1 y = (\gamma\epsilon p + \delta\eta q)y = \gamma\epsilon py + \delta\eta qy < \gamma + \delta = 1$$

but this is a contradiction, $L(r_1)$ being a support of C_1.
(the same argument holds for δ.)
Hence we have $\gamma > 0$, and $\delta > 0$, and this implies $\alpha = \gamma\epsilon > 0$, $\beta = \delta\eta > 0$ and $\alpha + \beta = \gamma\epsilon + \delta\eta \leqslant 1$.

This proves the theorem in R^2. Fig. 4.5.12 gives an illustration in X and P. Note that in the case of this figure the theorem holds, however the function $\mu(\lambda)$ is not maximum with respect to the point t_1, as it was assumed in the proof and as it was shown in fig. 4.5.11. Obviously, the theorem is also true in a point where $\mu(\lambda)$ takes on a maximum.
We now prove the theorem for $X = R^n$.

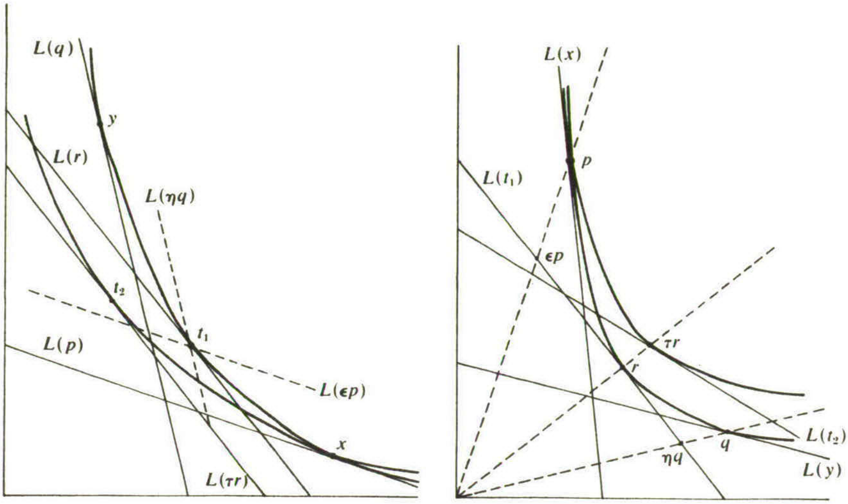


fig. 4.5.12

2. Let $n > 2$.
Since $C_1 \subset C_2$, by theorem 4.4.31, $C_1^* \supset C_2^*$, with
$p \in \text{Bnd}\, C_1^* \cap \text{Bnd}\, C_2^*$ and $q \in \text{Bnd}\, C_1^*$ and $q \notin C_2^*$.
We choose two vectors s_1 and s_2, such that:

$$s_1 = \gamma p - q \quad \text{where} \quad \gamma = \max\{\gamma' \mid \gamma' p - q \geqq 0\}$$
$$s_2 = -p + \delta q \quad \text{where} \quad \delta = \max\{\delta' \mid -p + \delta q \geqq 0\}$$

hence s_1 and s_2 satisfy the conditions (4.5.2).
These vectors span a two-dimensional subspace of P, that contains p and q:

$$p = \hat{p}^1 s_1 + \hat{p}^2 s_2 \quad \text{for} \quad \hat{p} = (\hat{p}^1, \hat{p}^2) \in R_+{}^2$$
$$q = \hat{q}^1 s_1 + \hat{q}^2 s_2 \quad \text{for} \quad \hat{q} = (\hat{q}^1, \hat{q}^2) \in R_+{}^2$$

hence

$$p = \hat{p}S \text{ and } q = \hat{q}S \quad \text{for} \quad S = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

Now $\hat{X}$ and $\hat{P}$ are defined as in (4.3.5), and $\hat{C}_1$ and $\hat{C}_2$ are the images of C_1 and C_2 (see (4.5.5)), while $\hat{C}_1^*$ and $\hat{C}_2^*$ are their duals in $\hat{P}$.
The conditions of the theorem are fulfilled in $\hat{X}$:

a. $\hat{C}_1 \subset \hat{C}_2$
b. $\hat{x} = Sx \in \text{Bnd}\, \hat{C}_1 \cap \text{Bnd}\, \hat{C}_2$ and $L(\hat{p})$ supports $\hat{C}_1$ in $\hat{x}$
(since $\hat{p}\hat{x} = \hat{p}Sx = px = 1$ and $L(p)$ supports C_1)
c. $\hat{y} = Sy \in \text{Bnd}\, \hat{C}_1$ and $y \notin \text{Bnd}\, \hat{C}_2$ and $L(\hat{q})$ supports $\hat{C}_1$ in $\hat{y}$.
(since $\hat{q}\hat{y} = \hat{q}Sx = qx = 1$ and $L(q)$ supports C_1)

Now the first part of the proof of this theorem ensures the existence of a plane $L(\hat{r}_1)$ that supports $\hat{C}_1$ in a point $\hat{t}_1$, whereas for the support $L(\hat{r}_2)$ of $\hat{C}_2$:

$$\hat{t}_2 \in L(\tau\hat{r}_1) \cap \text{Bnd}\,\hat{C}_2 \Rightarrow \hat{p}\hat{t}_2 \geqslant \hat{p}\hat{t}_1$$
$$\hat{r}_1 = \alpha\hat{p} + \beta\hat{q} \text{ with } \alpha > 0, \beta > 0 \text{ and } \alpha + \beta \leqslant 1$$

and we have

$$\hat{r}_1 \in \text{Bnd}\,\hat{C}_1^* \text{ and } \tau\hat{r}_1 \in \text{Bnd}\,\hat{C}_2^*.$$

If $r_1 = \hat{r}_1 S$, by theorem 4.5.8

$$r_1 \in \text{Bnd}\,C_1^* \text{ and } \tau r_1 \in \text{Bnd}\,C_2^*$$

and hence $L(r_1)$ supports C_1 and $L(\tau r_1)$ supports C_2.
Therefore, there exists t_1 such that

$$t_1 \in L(r_1) \cap \text{Bnd}\,C_1, \text{for } \hat{t}_1 = St_1$$

and

$$v \in L(r_1) \cap \text{Bnd}\,C_2 \Rightarrow pv \geqslant 1$$

since

$$\hat{v} = Sv \in L(\hat{r}_1) \cap \text{Bnd}\,\hat{C}_2$$

and therefore

$$pv = \hat{p}Sv = \hat{p}\hat{v} \geqslant 1$$

Obviously

$$r_1 = \alpha p + \beta q = \alpha\hat{p}S + \beta\hat{q}S = \hat{r}_1 S$$

4.6 SOME PROPERTIES OF REAL VALUED FUNCTIONS

A real valued function is a mapping of some set X into the real numbers (f: $X \to R$, see section 2.3). If on X a distance is defined, it is possible to define continuity with the help of this distance. We assume that $X \subset R^n$.

Definition 4.6.1

Let $X \subset R^n$ and u: $X \to R$. The function u is said to be *continuous in a point* $x \in X$, if

$$\forall\eta > 0, \exists\epsilon > 0: y \in B_\epsilon(x) \Rightarrow |u(x) - u(y)| < \eta$$

If this is true for every point $x \in X$, the function is called *continuous*. Hence continuity requires that given any positive number η, it is always possible to find a number ϵ, such that for any point y in the neighbourhood $B_\epsilon(x)$, the distance between $u(x)$ and $u(y)$ is smaller than η. A continuous real valued function always takes on extreme values on a closed and bounded set:

Theorem 4.6.2

If $u\colon X \to R$ is a continuous function and if $C \subset X \subset R^n$ is closed and bounded,

$$\exists \alpha \in R, \exists \beta \in R\colon \alpha = \min_{x \in C} u(x) = u(x_1) \text{ and } \beta = \max_{x \in C} u(x) = u(x_2)$$

The theorem, which we do not prove, means that the set $\{u \in R \mid \exists x \in C\colon u = u(x)\}$ has a greatest and a smallest element with respect to the relation $\geqslant$ on R.
Continuity can also be defined for mappings of R^n into R^m. For $f\colon R^n \to R^m$, it is required that

$$\forall \eta > 0, \exists \epsilon > 0\colon y \in B_\epsilon(x) \Rightarrow f(y) \in B_\eta(f(x))$$

Different continuity concepts are defined for correspondences (See e.g. Berge (1959).)
Finally, we define concave and convex functions:

Definition 4.6.3

A function $u\colon X \to R$, where $X \subset R^n$, is called *concave* if

$$\forall x, y \in X, \forall \lambda \in [0,1]\colon \lambda u(x) + (1-\lambda)u(y) \leqslant u(\lambda x + (1-\lambda)y)$$

Definition 4.6.4

A function $u\colon X \to R$, where $X \subset R^n$, is called *quasi concave* if

$$\forall x, y \in X, \forall \lambda \in [0,1]\colon \min\{u(x), u(y)\} \leqslant u(\lambda x + (1-\lambda)y)$$

If in definitions 4.6.3 and 4.6.4, $\leqslant$ is replaced by $\geqslant$, the functions are said to be convex and quasi convex respectively.

5. A consumer preference model

5.1 INTRODUCTION

The theory of consumer choice has been the subject of numerous publications in economics.[1] This interest of economists has probably two sources. In the first place, the satisfaction of the needs (in a wide sense) of consumers can be considered as the first, if not the only end of any economic process. In the second place, every economic model requires a representation of demand, and demand itself is in a way the reflection of the wants and needs of individuals. The question, if some economic system works satisfactorily, can only be answered if one knows whether the needs of individuals are met satisfactorily, where we leave it an open question what 'satisfactorily' means.

The first economists who were interested in consumer choice, took demand functions as their point of departure. Later on, demand functions were based on utility functions, which were first interpreted 'cardinally' and next 'ordinally'. The foundation of utility functions on a set of axioms with respect to preference relations is a relatively recent development.

In this and the next chapter, two axiomatic consumer choice models are considered, which can be interpreted in such a way that they treat the following case:

A consumer disposes in a certain relatively short period (week, month, year) of a certain amount of money, his disposable income $\tilde{m}$. There exist n different commodities with given prices $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n$. Given prices and income, the consumer chooses (buys) a certain non negative

1. See e.g. WOLD (1943), p. 85, SAMUELSON (1947) and the survey of HOUTHAKKER (1961).

quantity of each commodity, the combination (commodity bundle) being denoted $(x_1, x_2, \ldots, x_n)$.
In this chapter a model is considered, that is a special case of the preference model of chapter III. It is based on a binary relation between commodity bundles.
After having introduced the primitive concepts of the model, we present and discuss the set of axioms. Next, the utility function and the demand function are considered. Then the properties of the model in the commodity space (the set of commodity bundles) are deduced, and we show that in price space similar properties hold. Finally, we consider revealed preference and introduce a new revealed preference concept.

Remark

In discussing the models we shall adopt the interpretation given above. It is however possible to give them a slightly different interpretation. Let a consumer, at the beginning of a planning period consisting of k years, dispose of a certain amount of money $\tilde{m}_0$, whereas he will receive amounts $\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_k$ at the end of each year. Let the present value of these amounts, discounted at an interest rate r, be $\tilde{m}$, where

$$\tilde{m} = \tilde{m}_0 + \sum_{i=1}^{k} \tilde{m}_i \left(\frac{1}{1+r}\right)^i$$

There are l different commodities. The numbers $x^1, x^2, \ldots, x^l$ denote the quantities of commodities bought in the first period; $x^{l+1}, x^{l+2}, \ldots, x^{2l}$ are bought in the second period etc., hence x^{kl} is the quantity of the last commodity bought in the last period, and $kl = n$. The numbers $\tilde{p}^i$ $(i = 1, 2, \ldots, n)$ denote the discounted prices, hence the price that really will be paid, e.g. for $i = (k-1)l$, is $\tilde{p}^{(k-1)l}\,(1+r)^{k-1}$.

5.2 PRIMITIVE CONCEPTS

Because the consumer-choice models are special cases of the choice models that were discussed in chapter III, the same primitive concepts reappear, to which one new concept is added.

$X \subset R_+{}^n$ (choice space)
$\tilde{P} \subset R_+{}^{n+1}$ (price-income space)
$\gtrsim$ on X (preference relation)
$\mathscr{P}$ (set of choice sets)
$K: \mathscr{P} \to X$ (choice function)

There are n different commodities. Let X^i $(i = 1, 2, \ldots, n)$ be the set of all possible quantities of commodity i. The elements of X^i are denoted by x^i, hence $x^i \in X^i$. Since only non-negative quantities are possible, $x^i \geqq 0$, and hence $X^i \subset R_+$. The set X contains all possible combinations of the n commodities, that is, all possible *commodity bundles*

$$X = \prod_{i=1}^{n} X^i$$

X is the cartesian product of all X^i. Every element of X is a vector with n components and x^i is the quantity of commodity i: $x = (x^1, x^2, \ldots, x^n)$. X is a subset of the non-negative part of n-dimensional euclidean space: $X \subset R_+{}^n$. It is not necessary that every $x \in X$ is eligible from some choice set $M \in \mathscr{P}$. Now X' is the set of commodity bundles for which this is the case:

Definition 5.2.1

$$X' = \{x \in X \mid \exists M \in \mathscr{P}: x \in K(M)\}$$

The price-income space can be constructed in a similar way. Let $\tilde{P}^1, \tilde{P}^2, \ldots, \tilde{P}^n$ be the sets of values that the prices of the n commodities can have, and let $\tilde{P}^{n+1}$ be the set of all possible disposable money-incomes, hence $\tilde{p}^i \in \tilde{P}^i$ and $\tilde{m} \in \tilde{P}^{n+1}$. Then

$$\tilde{P} = \prod_{i=1}^{n+1} \tilde{P}^i$$

is the set of all possible combinations of prices and incomes; since prices and income are always non-negative,

$$\tilde{P} \subset R_+{}^{n+1}$$

Every point $\tilde{p} \in \tilde{P}$ is an $n+1$-vector $\tilde{p} = (\tilde{p}^1, \tilde{p}^2, \ldots, \tilde{p}^n, \tilde{m})$.
Henceforward, we shall also use another price-concept: If $\tilde{p} = (\tilde{p}^1, \tilde{p}^2, \ldots, \tilde{p}^n, \tilde{m}) \in \tilde{P}$, then $p = (p^1, p^2, \ldots, p^n) \in P$, where

$$p^i = \frac{\tilde{p}^i}{\tilde{m}} \; (i = 1, 2, \ldots, n)$$

The points of $\tilde{P}$ are absolute prices, expressed in money, with income as the last component, whereas the elements of P are 'relative' prices, expressed in disposable income. The income-component can now be dropped, since it is always equal to 1.

Definition 5.2.2

$$P = \left\{ (p^1, \ldots, p^n) \mid \exists \tilde{p} \in \tilde{P}: p^i = \frac{\tilde{p}^i}{\tilde{m}} \text{ for } (i = 1, \ldots, n) \right\}$$

Elements $p \in P$ correspond with all those elements $\tilde{p} \in \tilde{P}$, for which $\lambda > 0$ can be found, such that

$$\tilde{p} = \lambda(p^1, p^2, \ldots, p^n, 1) = (\lambda p^1, \lambda p^2, \ldots, \lambda p^n, \lambda).$$

P will be called *price-space*.

Remark

The set X contains elements, which a consumer could eventually *choose*. The points of $\tilde{P}$ (and of P) represent price-income situations, which eventually can occur.

With $p \in P$ corresponds by definition 4.4.3, a set $M(p) = \{x \in X \mid px \leq 1\}$. This set contains all commodity bundles that can be bought at the (relative) price p, since their value does not exceed the amount 1, (i.e. does not exceed the income $\tilde{m}$ at prices $\tilde{m}p$). The sets $M(p)$ will be called *budget sets*. The sets $L(p)$ of bundles that cost exactly 1 will be called *budget planes*. If $M(p) \in \mathscr{P}$ then this budget set is a choice set, and hence $M(p)$ represents a possible choice situation for the consumer. Therefore, we introduce a new set $P' \subset P$.

Definition 5.2.3

$$P' = \{p \in P \mid M(p) \in \mathscr{P}\}$$

The remaining concepts are identical to the ones discussed in 3.2: $\gtrsim$ is a preference relation between commodity bundles, $\mathscr{P}$ gives all sets of commodity bundles that can be choice sets and K is the choice function, which assigns the eligible commodity bundles from a choice set.

5.3 THE AXIOMS OF THE CONSUMER PREFERENCE MODEL

This model is characterised by ten axioms.[2] Three of these also occur in the preference model. Because all axioms of model P are also valid in the consumer model, model C is a special case of model P.

$C1$ (Extent of the commodity space)

$$X = R_+{}^n$$

$C2$ (Extent of the price space)

$$P = R_+{}^n$$

$C3$ (Transitivity)

$$x \gtrsim y \wedge y \gtrsim z \Rightarrow x \gtrsim z$$

$C4$ (Completeness)

$$\forall x, y \in X: x \gtrsim y \vee y \gtrsim x$$

$C5$ (Monotonicity)

$$x \geqq y \Rightarrow x \gtrsim y$$
$$x > y \Rightarrow x \succ y$$

$C6$ (Weak satiation)

$$x \sim x + t \wedge t \geqq 0 \Rightarrow$$
$$\forall \epsilon > 0, \exists \lambda > 0: [y \in B_\epsilon(x) \Rightarrow y + \lambda t \sim y + (\lambda + 1)t]$$

$C7$ (Continuity)

$$x \gtrsim y \wedge y \gtrsim z \Rightarrow \exists \alpha: 0 \leqslant \alpha \leqslant 1 \wedge y \sim \alpha x + (1-\alpha)z$$

$C8$ (Convexity)

$$\forall x_0 \in X: \{x \mid x \gtrsim x_0\} \text{ is convex}$$

$C9$ (Transition axiom)

$$\forall M \in \mathscr{P}: H(M) = K(M)$$

$C10$ (Extent of $\mathscr{P}$)

$$\mathscr{P} = \{M \mid \exists p \in P: M = M(p) \wedge H(M) \neq \varnothing\}.$$

These axioms will now be considered consecutively

2. Similar sets of axioms are given by WOLD (1945), § 30. WOLD (1953), chapter 4, DEBREU (1959), chapter 4. UZAWA (1960), p. 131.

Axioms C1 and C2

$$X = R_+^n, P = R_+^n$$

As x^i is a quantity of the i th commodity, x^i must be a non negative real number:

$$x^i \in R_+$$

This leaves unanswered the question which real values x^i can take. In principle, of many commodities only a countably infinite number of quantities are possible, e.g. the whole numbers. If x^3 is the number of cigarettes in the commodity bundle, obviously x^3 cannot be equal to 0.357 or to π. However, the set of whole numbers is rather difficult to manipulate. Therefore, it is usual to assume that all real numbers are permitted. In most cases this is not a very serious abstraction, since any real number is close to some whole number. Only for commodities of which no consumer buys more than a few items, like cars and T.V. sets, the real numbers are a very rough approximation. (By introducing different qualities of the same commodity it might be possible to interpret non whole numbers.[3]) By making the abstraction of axiom $C1$, we are able to assign on X the properties of R_+^n (see 4.2):

$$x, y \in X \Rightarrow x+y \in X$$

which now means: the sum of two commodity bundles is a commodity bundle, and

$$x \in X \wedge \lambda \in R_+ \Rightarrow \lambda x \in X$$

and this means: multiplication of a commodity bundle with a non negative real number gives another commodity bundle.

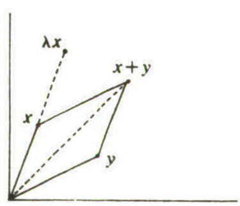


fig. 5.3.1

Similar remarks can be made with respect to P, though the abstraction is far less rigorous. Strictly, prices can only be expressed in whole cents.

3. See. e.g. HOUTHAKKER (1951).

Axioms C3 and C4

$C3 \quad x \succsim y \wedge y \succsim z \Rightarrow x \succsim z$
$C4 \quad \forall x, y: x \succsim y \vee y \succsim x$

These axioms require that the binary relation $\succsim$ is a complete preordering: the consumer is able to compare every couple of commodity bundles and the preference relation is transitive. (see definition 2.2.19 and section 3.3)

Axiom C5 (Monotonicity)[4]

$x \geqq y \Rightarrow x \succsim y$
$x > y \Rightarrow x \succ y$

The first part of this axiom requires that the partial ordering $\geqq$ on X coincides with the complete preordering $\succsim$.
Suppose a consumer possesses a certain commodity bundle. Would he like to get more of one particular commodity? This is not certain, as it is possible that his needs for this commodity are satiated. But, would he be willing to accept more? The answer to this question will be positive if he is able to eliminate freely any surplus of the commodity. If we suppose that this is possible, we can state that a bundle containing more of at least one commodity is at least as good as the original bundle.
The second part of the axiom excludes complete satiation: the consumer always prefers strictly bundles that contain more of each commodity. A sufficient condition for this is the existence of one commodity of which the consumer always wants to have more. Hence the relation $>$ coincides with the strict preference relation $\succ$.

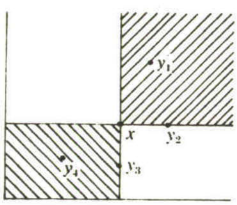


fig. 5.3.2

Figure 5.3.2 illustrates the axiom in R^2: the shaded areas represent the points that are greater and smaller respectively, than the point x.

4. In Wold (1943) occurs a weaker version of this axiom.

By the axiom we have

$$y_1 \succ x, y_2 \succsim x, x \succsim y_3 \text{ and } x \succ y_4$$

In the non shaded area, none of the relations $\geqq$ and $\leqq$ holds with respect to x. The axiom directly implies that equivalent points can never be in the interior of the shaded area:

$$x \sim y \Rightarrow (x-y) \not> 0 \wedge (x-y) \not< 0 \tag{5.3.3}$$

Axiom C6

$$x \sim x+t \wedge t \geqq 0 \Rightarrow$$
$$\forall \epsilon > 0, \exists \lambda > 0: [y \in B_\epsilon(x) \Rightarrow y+\lambda t \sim y+(\lambda+1)t]$$

To a bundle x is added a bundle t, and yet the bundle $x+t$ is not better than x. This is possible only if $t \not> 0$, for otherwise by $C5$, $x+t \succ x$.
The axiom states: if addition of some bundle t to some bundle x is not appreciated, then after addition of some multiple of t to any bundle, a new addition of t is not appreciated either. The size of λ, the multiple of t that has to be added first, depends on the distance between the point x and the arbitrary point y.
Applying $C8$ it will be shown that in fact

$$y+\lambda t \sim y+(\lambda+1)t \Rightarrow \forall \mu > 0: y+\lambda t \sim y+(\lambda+\mu)t$$

hence addition of any multiple of t to $y+\lambda t$ is no longer appreciated. It can be said that the consumer is satiated with the bundle t, given the bundle x.
This axiom has been included in order to avoid certain complications that might arise if price vectors occur with zero components. Apart from that, the axiom seems very plausible.

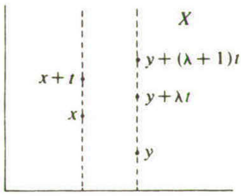


fig. 5.3.4

Axiom C7 (Continuity)

$$x \succsim y \wedge y \succsim z \Rightarrow \exists \alpha: (0 \leqslant \alpha \leqslant 1) \wedge y \sim \alpha x+(1-\alpha)z.$$

In words: if some point y is at least as good as a point x and not better than a point z, then on the line segment connecting the two points lies a point that is equivalent to y (see fig. 5.3.5 and 5.3.6).

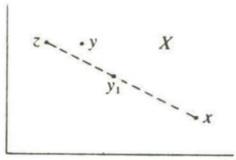


fig. 5.3.5

This axiom directly implies that the line segment connecting a point of the set $\{y \in X \mid y \geqq x\}$ with a point of $\{y \in X \mid y \leqq x\}$ contains a point that is equivalent to x. (See fig. 5.3.6, where $x_1 \sim x \sim x_2$.)

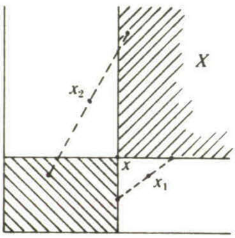


fig. 5.3.6

This entails that on every half line from the origin every equivalence class is represented, and it can be deduced now that a continuous utility function exists.

In literature,[5] another formulation frequently occurs, a formulation that is more general since it is meaningful in every preordered topological space.

With every $x_0 \in X$ can be associated the set $\{x \in X \mid x \gtrsim x_0\}$ of points that are preferred to x_0 (also called the *preference set* of x) and the set $\{x \in X \mid x_0 \gtrsim x\}$, the set of points to which x_0 is preferred. Now, instead of Axiom $C7$ one may require that both sets are *closed* (see definition 4.2.9.)

Theorem 5.3.7

Given axioms $C1$, $C3$, $C4$ and $C5$:

$$C7 \Leftrightarrow \forall x_0\colon \{x \in X \mid x \gtrsim x_0\} \text{ and } \{x \in X \mid x_0 \gtrsim x\} \text{ are closed.}$$

5. See DEBREU (1954).

Proof

Let $C_1 = \{x \mid x \gtrsim x_0\}$ and $C_2 = \{x \mid x_0 \gtrsim x\}$

1. $\Rightarrow$

We prove that $X - C_1$ and $X - C_2$ are *open*.
Let $y \in X - C_1$, hence $x_0 \succ y$.
Choose $z > x_0 + y$, hence $z > x_0$, $z > y$ and $z \succ x_0$.
By $C7$, α exists such that

$t = \alpha y + (1-\alpha)z$ and $t > 0$ and $t - y > 0$.

If one chooses $\epsilon < 1/n \min\limits_{1 \leq i \leq n} |t^i - y^i|$, then

for $v \in B_\epsilon(y)$ holds $t > v$ and hence $t \succ v$.
Therefore $B_\epsilon(y) \subset X - C_1$, hence y is an interior point of $X - C_1$, and this set is open.

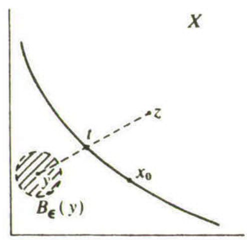


fig. 5.3.8

Let $y \in X - C_2$, hence $y \succ x$.
Since $x \gtrsim 0$, α exists such that

$x \sim z = \alpha y + (1-\alpha)0 = \alpha y$, $\alpha \in [0, 1]$

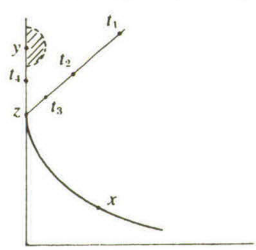


fig. 5.3.9

If $y > 0$, we have $y = (1/\alpha)z$ and $1/\alpha > 1$ and hence by Lemma 4.3.8, ϵ exists such that

$v \in B_\epsilon(y) \Rightarrow v > z$

hence $B_\epsilon(y) \subset X - C_2$ and this set is open.
If $y \not> 0$, choose $t_1 > y$. Let $y \sim t_2 = \beta z + (1-\beta)t_1$ and $t_3 = \frac{1}{2}z + \frac{1}{2}t_2$, hence $y \sim t_2 \succ t_3 \succ z$.
Now choose $t_4 = \gamma z + (1-\gamma)y$ such that $t_4 \sim t_3$.
Hence $y \geq t_4 \geq z$ and therefore by Lemma 4.3.8, ϵ exists such that $v \in B_\epsilon(y) \Rightarrow v \geq t_4$, hence $v \succ z$ and this implies $B_\epsilon(y) \subset X - C_2$.

2. $\Leftarrow$

Let $y \in C_1$ and $z \in C_2$, where both sets are closed.
Define

$$T = \{v \in X \mid \exists \alpha \in [0,1]: v = \alpha y + (1-\alpha)z\}$$

and this set is closed and convex.
The sets $T_1 = T \cap C_1$ and $T_2 = T \cap C_2$ are also closed and convex and $T = T_1 \cup T_2$.
Hence by theorem 4.2.23

$$T_1 \cap T_2 \neq \varnothing$$

and the points of this intersection are equivalent to x_0.

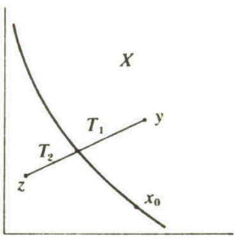


fig. 5.3.10

This theorem implies that the set $\{x \in X \mid x \sim x_0\}$, the equivalence class of x_0, is also closed, since it is the intersection of two closed sets.
To illustrate the meaning of the axiom we give two examples where it is not true.

Example 5.3.11

Let $x_1 \sim x \sim x_3$ and S is the shaded area, including the drawn line, but without the points x_1, x_2 and x_3.
T is the rest of $X = R_+{}^2$:

$S = \{x \mid x_1 \succ x\}$ and
$T = \{x \mid x \succ x_1\}$

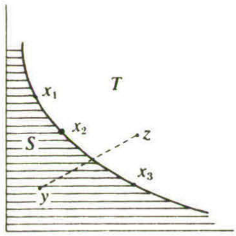


fig. 5.3.12

Now $y \in S$ and $z \in T$, but the line segment connecting these two points does not contain any point that is equivalent to x_1.

Example 5.3.13

A well known case[6] of an ordering where the continuity axiom is not true is the *lexicographic ordering*. In this case all other axioms of model C hold. We shall consider this ordering for $X = R_+{}^2$.
Between two vectors $x = (x^1, x^2)$ and $y = (y^1, y^2)$ the strict preference relation $\succ$ holds in two cases:

$x \succ y \Leftrightarrow [(x^1 > y^1) \vee (x^1 = y^1 \wedge x^2 > y^2)]$

The first component of the vector is considered first: the vector with the greatest first component is the best, whatever the second components may be. Only if the first components are equal the second components are considered, and now the vectors are ranked according to this component.

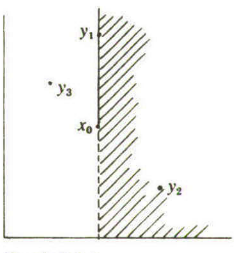


fig. 5.3.14

In fig. 5.3.14 we have

$y_1 \succ x_0, y_2 \succ x_0, x_0 \succ y_3$

The space $X = R_+{}^2$ can be partitioned into the sets $\{x \mid x \succsim x_0\}$ and $\{x \mid x_0 \succsim x\}$.

6. See DEBREU (1954). CHIPMAN (1960).

None of these sets is closed, since the first set contains the shaded area (fig. 5.3.14), including the drawn line, but excluding the dotted line, (below x_0), whereas the second set consists of the non shaded area, including the dotted line but without the drawn line.
Obviously, on the line segment connecting two points x_1 and x_2, with $x_1 \succ x_0 \succ x_2$, no point lies that is equivalent to x_0, unless x_0 itself is on this line. There exists no other point that is equivalent to x_0:

$$\{x \mid x_0 \gtrsim x_0\} \cap \{x \mid x_0 \gtrsim x\} = \{x_0\}$$

Hence the binary relation is a complete *ordering* (see definition 2.2.22).

Axiom C8 (Convexity)

$$\forall x_0 \in X: \{x \in X \mid x \gtrsim x_0\} \text{ is convex}$$

Now we can state:

Theorem 5.3.15

$$\forall x_0 \in X: \{x \in X \mid x \gtrsim x_0\} \text{ is a closed c.u.p. set}$$

Proof

Convexity by $C8$, unboundedness by $C5$, and by $C1$, $X = R_+{}^n$. By theorem 5.3.7 the set is closed.

Convexity requires that all points on a line segment connecting two points that are preferred to a third, are also preferred to this point, or equivalently, all points on a line segment are preferred to the worst of the two points it connects.

$\forall x, y \in X$:

$$\lambda \in [0,1] \Rightarrow z = \lambda x + (1-\lambda)y \gtrsim x \vee z = \lambda x + (1-\lambda)y \gtrsim y$$

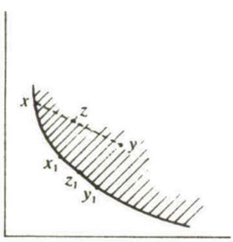


fig. 5.3.16

In fig. 5.3.16 the shaded area is the preference set of x_0. Since $y \gtrsim x_0$, we have $z = \lambda x + (1-\lambda)x_0 \gtrsim x_0$ (for $\lambda \in [0, 1]$), and since $x_1 \gtrsim x_0$ and $y_1 \gtrsim x_0$, we have $z_1 = \lambda x_1 + (1-\lambda)y_1 \gtrsim x_0$ (for $\lambda \in [0, 1]$). If $x \sim y$ and $v = x - y$, we have $v \ngtr 0$ and $v \nless 0$, as already shown in (5.3.3). If e.g. $v^i > 0$ for $i = 1, 2, \ldots, k$ and $v^i < 0$ for $i = k+1, k+2, \ldots, n$, this means that the consumer is willing to offer a certain quantity of the first k commodities in exchange for a certain quantity of the last $n-k$ commodities. If $y = x + v + \rho x$, convexity implies that the consumer is not willing to exchange more in the same proportions.

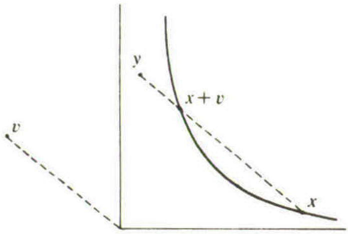


fig. 5.3.17

It appears that this is equivalent to the axiom. If we write $y = x + v + \rho v$ and $\lambda = 1/(1+\rho)$, we can state

Theorem 5.3.18

Given axioms $C1$, $C3$, $C4$, $C5$, $C7$:

$C8 \Leftrightarrow \; [((x \sim z) \wedge z = \lambda x + (1-\lambda)y) \Rightarrow x \gtrsim y]$

1. Let the axiom be true, hence $\{t \mid t \gtrsim x\}$ is convex and we have $x \sim z$ and $z = \lambda x + (1-\lambda)y$. Suppose $y \succ x$. By $C7$, y_1 must exist such that

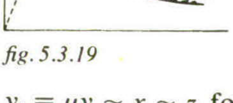


fig. 5.3.19

$y_1 = \mu y \sim x \sim z$, for $\mu < 1$.

Hence if

$x_1 = \lambda x + (1-\lambda)y_1$,

by the convexity axiom

$x_1 \gtrsim y_1 \sim x \sim z$

However since $x_1 < z$, also $z \succ x_1$ and that is a contradiction.
2. Let the statement be true and suppose the axiom is not. Now two points x and y exist such that $y \gtrsim x$, and a number $\mu \in [0, 1]$ such that (see fig. 5.3.20)

$t = \mu x + (1-\mu)y$ and $x \succ t$.

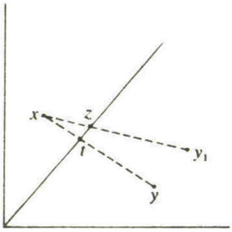


fig. 5.3.20

Choose $z \sim x$, with $z = \lambda t$, for $\lambda > 1$.
If $\lambda t = z = \mu x + (1-\mu)y_1$,
then we have

$(\lambda - 1)t = (1-\mu)(y_1 - y)$

and this implies $y_1 - y > 0$, hence $y_1 \succ y$ and $y_1 \succ x$ and that is a contradiction, for we have

$x \sim z$ and $z = \mu x + (1-\mu)y_1$ and this implies $x \gtrsim y_1$.

These two equivalent formulations establish the intuitive base of the convexity axiom. It seems a reasonable assumption that a convex combination of two bundles is not worse than each one, but verification is very difficult.[7]
A case as shown by fig. 5.3.21 is not inconceivable.

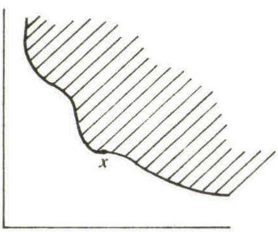


fig. 5.3.21

7. An experiment of the psychologist Thurstone produced some empirical support for the axiom. See THURSTONE, chapter XII, p. 123.

Finally, we prove the result announced during the discussion of axiom C6.

Theorem 5.3.22

$$x \sim x+t \wedge t \geq 0 \Rightarrow \forall \mu > 0: x \sim x+\mu t$$

Proof

If $0 < \mu < 1$ we have $x \leq x+\mu t \leq x+t$, hence
$x \sim x+t \gtrsim x+\mu t \gtrsim x$, hence $x \sim x+\mu t$.
Let $\mu > 1$.
Since $x+\mu t \geq x$, we have $x+\mu t \gtrsim x$. Choose $\alpha = 1/\mu \in [0, 1]$.
Now we have

$$x \sim x+t = \alpha(x+\mu t)+(1-\alpha)x$$

and by theorem 5.3.18 this implies

$$x \gtrsim x+\mu t$$

Hence

$$x \sim x+\mu t$$

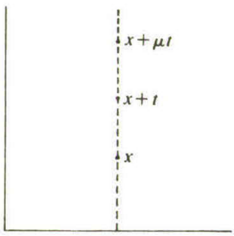


fig. 5.3.23

Axiom C9

$K(M) = H(M)$ for $M \in \mathscr{P}$.

This axiom is identical to $P4$: the eligible elements of a choice set are its greatest (best) elements.

Axiom C10

$$\mathscr{P} = \{M \subset X \mid \exists p \in P: M = M(p) \wedge H(M(p)) \neq \emptyset\}$$

This axiom states that the choice sets are the budget sets that have a greatest element (with respect to the binary relation $\gtrsim$). The last requirement is identical to axiom $P3$.
By applying $C9$ it follows

$$K(M(p)) \neq \varnothing.$$

By definition 5.2.3

$$p \in P' \Rightarrow M(p) \in \mathscr{P}.$$

Now from $C10$ follows

$$M(p) \in \mathscr{P} \Rightarrow p \in P',$$

hence combining the two above statements:

$$P' = \{p \in P \mid H(M(p)) \neq \varnothing\} \tag{5.3.24}$$

The choice function is only defined for budget sets. This implies that choice sets and hence choice, only depend on *relative* prices, since by definition 4.4.3

$$M(p) = \{x \in X \mid px \leqslant 1\} = \{x \in X \mid \tilde{p}x \leqslant \tilde{m}\}$$

If

$$p^i = \frac{\tilde{p}^i}{\tilde{m}},$$

we have

$$(\tilde{p}, \tilde{m}) = m(p, 1).$$

Hence absolute prices are irrelevant, for if

$$(\tilde{p}_2, \tilde{m}_2) = \lambda(\tilde{p}_1, \tilde{m}_1),$$

then holds

$$p_1{}^i = \frac{1}{\tilde{m}_1}\tilde{p}_1{}^i = \frac{1}{\tilde{m}_2}\tilde{p}_2{}^i = p_2{}^i, \; (i = 1, \ldots, n).$$

The axiom also requires that each commodity is sold at a price that is independent of the quantity consumed. This does not only exclude certain discounts, but also the possibility that at a fixed amount an unlimited quantity of some commodity can be obtained. If one commodity is sold at a price per unity (e.g. bread) and a second at a fixed amount (e.g. water), a choice set as shown in fig. 5.3.25 results. If only bread is consumed, the greatest quantity that can be bought, is x^1. If the consumer also wants water, he can at most buy y^1 of bread, but he can drink an unlimited quantity of water.

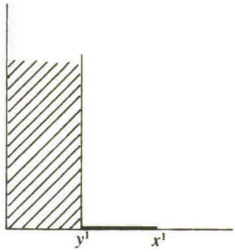


fig. 5.3.25

The present model is not adapted to this kind of problem. Note however that it is permitted that one or more commodities have price 0 (collective goods and so called free goods). If e.g. the price of water would be 0 (either as a collective or as a free good), a choice set in the water-and-bread case will look like fig. 5.3.26. The consumer can always drink as much water as he pleases, and he can buy x^1 of bread. Obviously, from the consumer's earnings the *compulsory* contribution to the Treasury (to finance water works among other things), has been subtracted first: $\tilde{m}$ is his disposable income.

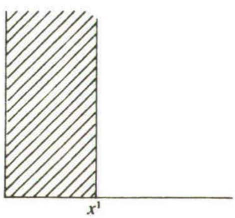


fig. 5.3.26

5.4 THE UTILITY FUNCTION

The axioms of the consumer preference model guarantee the existence of an order-preserving function, which is continuous.[8]

8. WOLD (1943), p. 223 and 259, and WOLD (1953), p. 83; DEBREU (1954) gives a more general proof.

Theorem 5.4.1

Given the axioms $C1$, $C3$, $C4$, $C5$ and $C6$ there exists a *continuous* function

$$u: X \to R$$

where $u(x) \geqslant u(y)$, if $x \gtrsim y$ and $u(x) > u(y)$, if $x \succ y$.

Proof

Choose an arbitrary point $z_0 \in X$, such that $z_0 > 0$.
Let Z be the set of all points on a half line from the origin through z_0:

$$Z = \{x \in X \mid \exists\lambda > 0: x = \lambda z_0\}.$$

In virtue of the monotonicity axiom, Z is completely ordered, for

$$[x = \lambda z_0 \wedge y = \lambda' z_0 \wedge x \sim y] \Rightarrow x = y$$

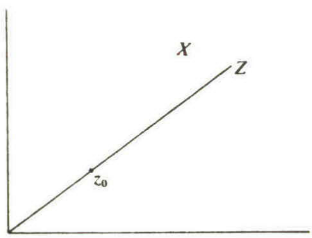


fig. 5.4.2

If we define the function $v: Z \to R$, such that

$$v(x) = \lambda \text{ if } x = \lambda z_0$$

then v is an order-preserving function of Z into R, where $v(z_0) = 1$ and $v(0) = 0$.
For every $x \in X$, a point $y \in Z$ exists such that $y \sim x$: two numbers $\lambda_1 > 0$ and $\lambda_2 > 0$ can be found such that

$$0 \leqq \lambda_1 z_0 \leqq x \leqq \lambda_2 z_0$$

and by the continuity axiom, $\alpha \in [0, 1]$ exists, for which

$$x \sim y = \alpha\lambda_1 z_0 + (1-\alpha)\lambda_2 z_0 = (\alpha\lambda_1 + (1-\alpha)\lambda_2) z_0,$$

hence $y \in Z$.

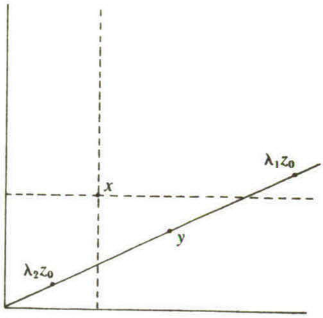


fig. 5.4.3

Now the function u: $X \to R$ can be defined:

$u(x) = \lambda$ if $x \sim \lambda z_0$

We still have to show that $u(x)$, as defined, is continuous (see definition 4.6.1), which requires that for every $x \in X$:

$\forall \epsilon, \exists \eta$: $d(x, y) < \eta \Rightarrow |u(x) - u(y)| < \epsilon$.

Choose ϵ arbitrarily. Let $u(x) = \lambda$, hence $x \sim \lambda z_0$.
Choose $\lambda_1 = \lambda + \epsilon$ and $\lambda_2 = \lambda - \epsilon$, such that

$\lambda + \epsilon = u(\lambda_1 z_0) > u(\lambda z_0) = \lambda = u(x) > u(\lambda_2 z_0) = \lambda - \epsilon$

The sets $\{t \mid \lambda_1 z \succ t\}$ and $\{t \mid t \succ \lambda_2 z\}$ are open, and so is their intersection, which contains x:

$x \in \{t \mid \lambda_1 z \succ t \wedge t \succ \lambda_2 z\}$.

Therefore, by definition 4.2.7 there exists a neighbourhood

$B_\eta(x) \subset \{t \mid \lambda_1 z \succ t \wedge t \succ \lambda_2 z\}$

hence

$y \in B_\eta(x) \Rightarrow \lambda - \epsilon < u(y) < \lambda + \epsilon \Rightarrow |u(x) - u(y)| < \epsilon$.

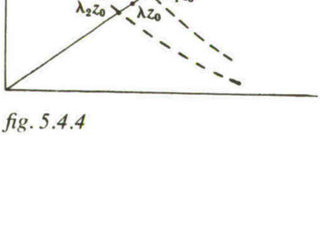


fig. 5.4.4

Clearly, it is also possible to construct a function that is not continuous, but a continuous one does in fact exist. Roughly speaking, continuity requires, that the utility index makes no 'jumps': it is not possible that the utility is 7 in some point and 9 in an adjacent point. If two points have the utilities 7 and 9, then the utility takes all values between 7 and 9 on the line segment joining the two points. It should be noted that the utility function needs not be differentiable.
By the monotonicity axiom, the utility function is nondecreasing.

Theorem 5.4.5

If $x, y \in X$

$$x > y \Rightarrow u(x) > u(y)$$
$$x \geqq y \Rightarrow u(x) \geqslant u(y)$$

Proof

$$x > y \Rightarrow x \succ y \Rightarrow u(x) > u(y)$$
$$x \geqq y \Rightarrow x \succsim y \Rightarrow u(x) \geqslant u(y)$$

In the proof of the existence of a continuous utility function we did not use the convexity axiom. This axiom implies that the utility function is *quasi-concave* (definition 4.6.4).

Theorem 5.4.6

If the axioms $C1$, $C3$, $C4$, $C5$, $C7$ and $C8$ are valid, every order preserving function on X is *quasi-concave*.

Proof

$x \sim y \Leftrightarrow u(x) = u(y)$, hence $u(x) = \lambda u(x) + (1-\lambda)u(y) = u(y)$.
By $C6$, $\lambda x + (1-\lambda)y \succsim x \sim y$, hence $u(\lambda x + (1-\lambda)y) \geqslant u(x)$.

Concavity is a stronger requirement. The utility function would then have a decreasing slope, as is illustrated in fig. 5.4.7.
If x is an arbitrary point and V is the half-line from the origin that passes through x, so that $\tau x \in V$ for every $\tau > 0$, then

$$\lambda u(x) + (1-\lambda)u(y) \leqslant u(\lambda x + (1-\lambda)y) \text{ for } y = \tau_0 x,\ \tau_0 > 0$$

For differentiable functions, this is equivalent to decreasing marginal utility. This property is not maintained under increasing transformations

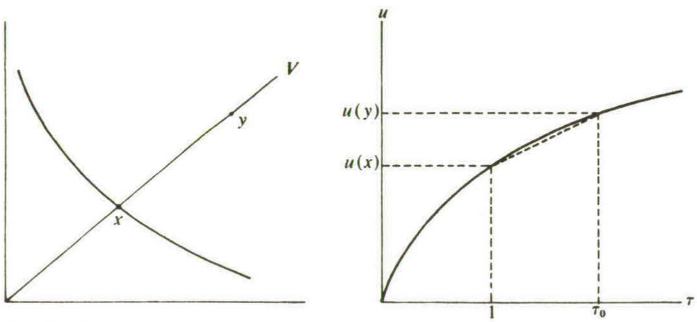


fig. 5.4.7

of the utility function and these transformations are not ruled out by our axioms. Concave utility functions would only be permitted, if utility differences were comparable (see 2.5), and this would require more axioms.[9]

5.5 THE DEMAND FUNCTION AND ITS CONNECTION WITH THE CHOICE FUNCTION

In virtue of (5.3.24)

$$p \in P' \Leftrightarrow M(p) \in \mathscr{P}$$

The choice function associates with every $M(p)$, where $p \in P'$, the eligible elements of $M(p)$, which are, because of axiom $C9$, best elements. We now introduce a correspondence $G: P' \to X'$, called *demand function* which associates eligible elements directly with prices:

Definition 5.5.1

$$\forall p \in P': G(p) = K(M(p))$$

By definition 3.5.1 choice sets are preordered by the binary relation $\gtrsim^*$. Because of definition 5.5.1, for every $M(p), M(q) \in \mathscr{P}$

$$M(p) \gtrsim^* M(q) \Leftrightarrow \exists x, y: x \in G(p) \wedge y \in G(q) \wedge x \gtrsim y \qquad (5.5.2)$$

The choice function can be replaced by the demand function, because only budget-sets can be choice-sets (by $C10$). It is quite plausible to apply the binary relation $\gtrsim^*$, which was defined for choice sets, to prices and to do the same with the other binary relations on $\mathscr{P}$. Instead

9. See e.g. DEBREU (1960), p. 20.

of $M(p) \succsim^* M(q)$, we shall write $p \succsim^* q$. Hence $p \succsim^* q$ (p is at least as favourable as q) means $M(p) \succsim^* M(q)$ (the set $M(p)$ is at least as favourable as the set $M(q)$).

Convention 5.5.3

Henceforward, the binary relations $\succsim^*$, R^*, R^{*k}, $\bar{\mathsf{R}}^*$ on $\mathscr{P}$ are replaced by the same relations on P'.

Definition 3.4.1 can now be written

$$p\mathsf{R}^*q \Leftrightarrow M(p) \cap G(q) \neq \varnothing \quad (\text{if } p, q \in P')$$

and this means: p is revealed at least as favourable as q, if and only if at price q, a commodity bundle is demanded, which could also be bought at price p.

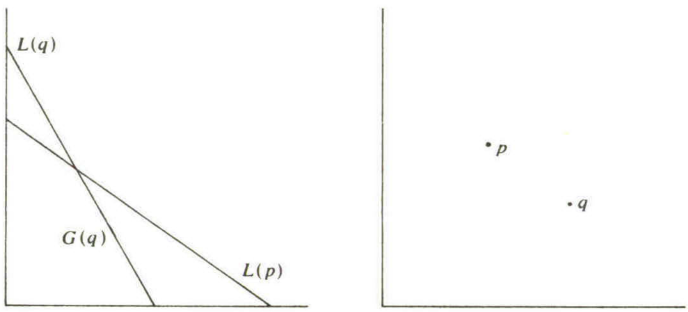


fig. 5.5.4

Since model C is a special case of model P, all theorems of 3.4 and 3.5 for the binary relations on $\mathscr{P}$, are also valid in this model, and because of Convention 5.5.3 they are also true for P'. Thus the strong property of revealed preference can now be formulated:

$$p\bar{\mathsf{R}}^*q \Rightarrow q\not{\mathsf{P}}^*p$$

All formulas of chapter III can be 'translated' in the following way:

Replace P, Q, etc. in the binary relations by p, q, etc. and elsewhere by $M(p), M(q)$, etc.

Replace $K(P), K(Q)$, etc. by $G(p), G(q)$, etc.

5.6 PREFERENCE SETS

The utility function is a mapping u: $X \to R$, which is continuous, non-decreasing and quasi-concave (theorems 5.4.1, 5.4.5 and 5.4.6).

The set $U \subset R$ is the set of all real numbers, which are utility index of some point of the choice space:

Definition 5.6.1

$$U = \{u \in R \mid \exists x \in X: u(x) = u\}$$

Now the utility function is a mapping of X *onto* U (see section 2.3). For the mapping u, as it was defined in theorem 5.4.1, U is the set of all non-negative real numbers, hence

$$U = \{u \in R \mid u \geqslant 0\} \tag{5.6.2}$$

In 4.3 we introduced the sets:

$$\{x \in X \mid x \gtrsim x_0\} \text{ and } \{x \in X \mid x_0 \gtrsim x\}$$

The first contains the points that are preferred to x_0, and the second is the set of points to which x_0 is preferred. Clearly, the elements of their intersection are indifferent to x_0. Since such sets are associated with every $x \in X$, we could consider them as the result of a correspondence of X into X. However, we can also define them by means of the utility function, as a correspondence C: $U \to X$:

Definition 5.6.3

If $u_0 \in U$,

$$C(u_0) = \{x \in X \mid u(x) \geqslant u_0\}$$

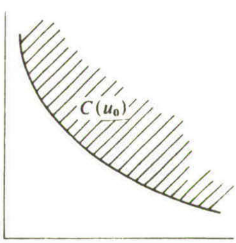


fig. 5.6.4

The set $C(u_0)$ contains all points with utilities of at least u_0. If $u(x_0) = u_0$,

$$C(u_0) = C(u(x_0)) = \{x \in X \mid x \gtrsim x_0\} \tag{5.6.5}$$

Hence $C(u_0)$ is the preference set of every commodity bundle with utility u_0, and therefore:

Theorem 5.6.6

For every $u_0 \in U, C(u_0)$ is a closed c.u.p. set.

Proof

By theorem 5.3.15, every preference set is a closed c.u.p. set.

If u decreases, the correspondence C associates with u, a decreasing series of sets.[10]

Theorem 5.6.7

$u_1, u_2 \in U \wedge u_1 > u_2 \Rightarrow C(u_1) \subset C(u_2).$

Proof

$x \in C(u_1) \Leftrightarrow u(x) \geqslant u_1 > u_2 \Rightarrow x \in C(u_2)$

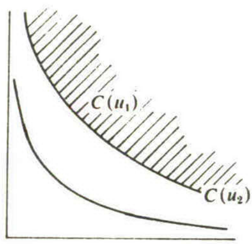
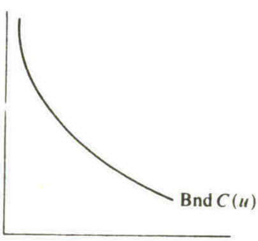


fig. 5.6.8

Now it follows:

$C(0) = X$ and $u > 0 \Rightarrow C(u) \neq X$

Every set $C(u)$ can be divided into two disjoint sets, namely the interior and the boundary of $C(u)$, and by theorem 4.3.10



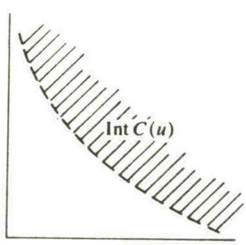


fig. 5.6.9

10. It can also be shown that the correspondence C is lower-semi-continuous.

$$\operatorname{Int} C(u) = \{x \in C(u) \mid \exists\lambda : 0 < \lambda < 1 \wedge \lambda x \in C(u)\}$$
$$\operatorname{Bnd} C(u) = \{x \in C(u) \mid \lambda < 1 \Rightarrow \lambda x \notin C(u)\}$$

Clearly,

$$\begin{aligned} &\operatorname{Int} C(u) \cap \operatorname{Bnd} C(u) = \varnothing \\ &\operatorname{Int} C(u) \cup \operatorname{Bnd} C(u) = C(u) \end{aligned} \tag{5.6.10}$$

Both Bnd $C(u)$ and Int $C(u)$ can be considered as correspondences of U into X. For the properties of the sets we refer to theorem 4.3.10. The sets Int $C(u)$ are, like $C(u)$, a decreasing series if u increases (see theorem 5.6.7) and[11]

$$u_1 \neq u_2 \Rightarrow \operatorname{Bnd} C(u_1) \cap \operatorname{Bnd} C(u_2) = \varnothing$$

Int $C(u)$ contains *certainly* all points with utilities larger than u, whereas Bnd $C(u)$ contains *nearly* all elements with utility u.

Theorem 5.6.11

For every $u \in U$

$$\operatorname{Bnd} C(u) \subset \{x \in X \mid u(x) = u\} \tag{a}$$
$$\operatorname{Int} C(u) \supset \{x \in X \mid u(x) > u\} \tag{b}$$

Proof

a. Let $x \in \operatorname{Bnd} C(u)$, hence $u(x) \geqslant u$. For every $\lambda < 1$, $\lambda x \notin C(u)$, hence $u(\lambda x) < u$. Since u is continuous, it follows $u(x) = u$.
b. Since $\operatorname{Int} C(u) = C(u) - \operatorname{Bnd} C(u)$,
$C(u) - \{x \in X \mid u(x) = u\} = \{x \in X \mid u(x) > u\} \subset \operatorname{Int} C(u)$.

Bnd $C(u)$ contains all points with utility u, with the exception of points $x \in X$, for which λ exists, such that

$$\lambda < 1 \text{ and } \lambda x \sim x$$

and hence

$$\lambda x \leq x \text{ and } \lambda x \sim x.$$

In virtue of the monotonicity axiom, this is only possible if x has at least one 0-component: $x^i = 0$, for some $i \in \{1, 2, \ldots, n\}$ (see fig. 5.6.12).

11. The correspondence Bnd $C(u)$ is also lower-semi-continuous.

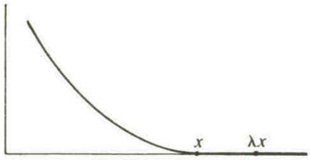


fig. 5.6.12

For strictly positive commodity bundles

$$x > 0 \wedge u(x) = u \Rightarrow x \in \operatorname{Bnd} C(u) \tag{5.6.13}$$

So every positive x must be a boundary point of the preference set $C(u(x))$.

5.7 DUALITY[12] IN CONSUMER CHOICE THEORY

With every $u_0 \in U$ corresponds a preference set $C(u_0)$ which is a closed c.u.p. set. By definition 4.4.12, with every preference set can be associated the dual set $C^*(u_0) \subset P$, where

$$\begin{aligned} C^*(u_0) &= \{p \in P \mid L(p) \cap \operatorname{Int} C(u_0) = \varnothing\} \\ &= \{p \in P \mid M(p) \cap \operatorname{Int} C(u_0) = \varnothing\} \end{aligned}$$

and by theorem 4.4.14, $C^*(u_0)$ is also a closed c.u.p. set. So $C^*(u_0)$ is the set of all prices at which *no commodity bundles* with utilities *higher* than u_0, can be bought. Taking into account definition 4.4.1, the above formula can be replaced by

$$C^*(u_0) = \{p \in P \mid x \in C(u_0) \Rightarrow px \geqslant 1\} \tag{5.7.1}$$

and also by (see definition 5.6.3)

$$C^*(u_0) = \{p \in P \mid x \in M(p) \Rightarrow u(x) \leqslant u_0\} \tag{5.7.2}$$

Clearly, C^* is a correspondence

$$C^*: U \rightarrow P$$

and has similar properties[13] as the correspondence C.

Theorem 5.7.3

$$u_1, u_2 \in U \wedge u_1 > u_2 \Rightarrow C^*(u_1) \supset C^*(u_2)$$

12. Duality was first applied in consumer choice theory by Roy, who introduced the dual utility function. See Roy.
13. It can also be shown that the correspondence C^* is lower-semi-continuous.

Proof

Since $u_1 > u_2$, it follows by theorem 5.6.7, $C(u_1) \subset C(u_2)$, hence the statement of the theorem is directly implied by applying theorem 4.4.31.

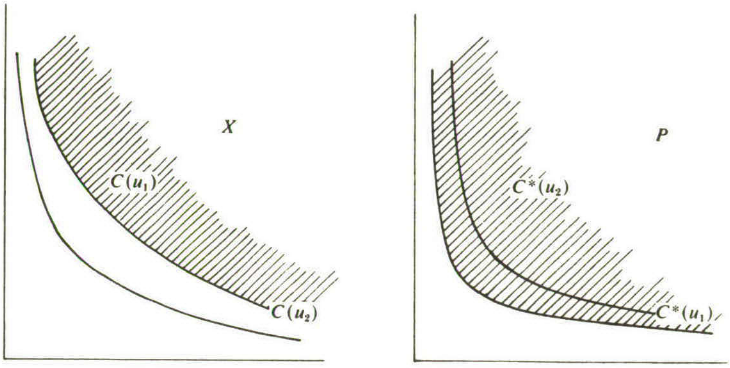


fig. 5.7.4

Theorem 5.7.3 can be read: if at some price, no commodity bundle with utility higher than u_2 can be obtained, then it is also impossible to get a bundle with utility higher than u_1 at this price, for $u_1 > u_2$. This is too evident!

If $p \in P$ and $u_0 \in U$, four cases can occur:

(a) $p \notin C^*(u_0)$

By definition 4.4.12 this implies

$$M(p) \cap \operatorname{Int} C(u_0) \neq \varnothing$$

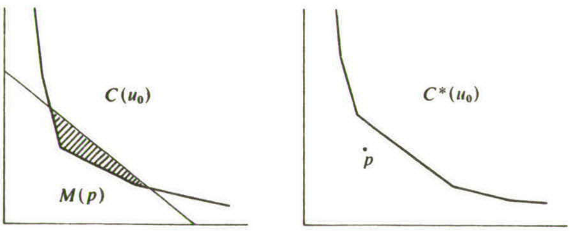


fig. 5.7.5

Now the budget set contains interior points of the preference set and, since $\operatorname{Int} C(u_0)$ is open, in the intersection lies also a strictly positive commodity bundle and thus $M(p)$ contains a bundle with a utility higher than u_0. Hence at price p, this utility is attainable.

(b) $p \in \operatorname{Int} C^*(u_0)$

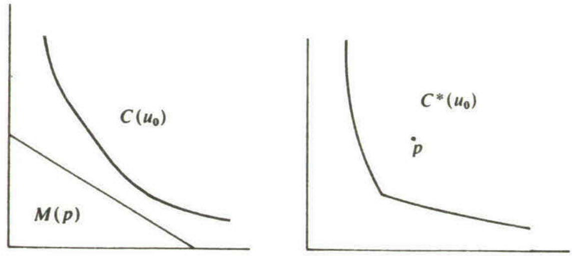


fig. 5.7.6

By theorem 4.4.21, we now have

$$M(p) \cap C(u_0) = \varnothing$$

So the budget set contains no point with a utility equal to u_0 or higher than u_0. It is impossible to buy such a bundle at price p, since it costs more than 1. (It costs more than $\tilde{m}$ at price $\tilde{p}$.)

(c) $p \in \operatorname{Bnd} C^*(u_0)$ and $M(p) \cap C(u_0) = \varnothing$

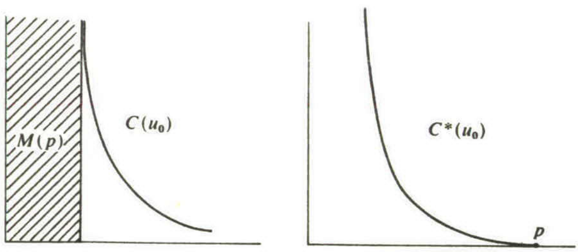


fig. 5.7.7

By theorem 4.4.18, this case can only occur if $p \not> 0$. Now the budget plane is asymptotic to the preference set, and the utility u_0 is just not attainable, though a proportional decrease of prices, however small, will make the two sets intersect.

(d) $p \in \operatorname{Bnd} C^*(u_0)$ and $M(p) \cap C(u_0) \neq \varnothing$.

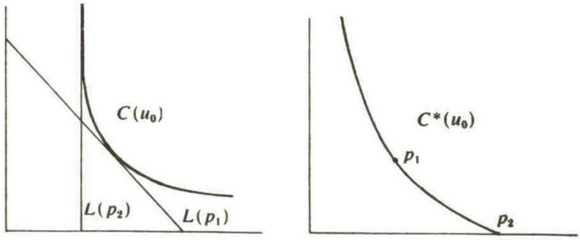


fig. 5.7.8

Now the budget plane touches the preference set, hence $L(p)$ is a supporting hyperplane. So $M(p)$ contains a boundary point $x_0 \in C(u_0)$ and by theorem 5.6.11, this point has a utility u_0. The utility u_0 is the highest attainable utility at price p.
Let us first consider case (d); we have

$$x_0 \in M(p_0) \cap \operatorname{Bnd} C(u_0) \wedge M(p_0) \cap \operatorname{Int} C(u_0) = \varnothing \tag{5.7.9}$$

and this is equivalent to (see (5.7.1))

$$u(x_0) = u_0 = \max_{x \in M(p_0)} u(x) \tag{5.7.10}$$

and also to

$$\min_{x \in C(u_0)} p_0 x = p_0 x_0 = 1 \tag{5.7.11}$$

(5.7.10) implies $x \in M(p_0) \Rightarrow u(x) \leq u_0$, hence $x \in M(p_0) \Rightarrow x_0 \gtrsim x$, so x_0 is a greatest element of $M(p_0)$:

$$x_0 \in H(M(p_0)) \neq \varnothing$$

and applying axiom $C10$, it follows that p P' and (see definition (5.5.1)

$$x_0 = K(M(p)) = G(p).$$

So, if for $p \in P$ a number u can be found, such that $L(p)$ touches the preference set $C(u)$, then we have $p \in P'$ and $G(p) \neq \varnothing$.
The set $P' \subset P$ contains anyhow all strictly positive prices, and it may also contain prices with one or more 0-components. If p is such a price, then in virtue of axiom $C6$, all prices that have 0-components which are also 0 in p, but no other ones, are in P' too. We shall also show that P' is convex.

Theorem 5.7.12

$$p > 0 \Rightarrow p \in P'$$

Proof

If $p > 0$, $M(p)$ is closed and bounded and since $u(x)$ is a continuous function, there exists a point x_0, such that

$$u(x_0) = \max_{x \in M(p)} u(x)$$

Now $L(p)$ touches the preference set $C(u(x_0))$ in x_0.

Theorem 5.7.13

$$[p \in P' \wedge (q^i = 0 \Rightarrow p^i = 0)] \Rightarrow q \in P'$$

Proof

The theorem is always valid for $p > 0$.
So, let $p^i = 0$ for one or more $i \in \{1, 2, \ldots, n\}$.
Since $p \in P'$, x exists such that $x \in G(p)$.
Choose t such that

$t^i > 0$ if $q^i = 0$ (and $p^i = 0$)
$t^i = 0$ if $q^i \neq 0$

then $p(x+t) = 1$, hence $(x+t) \in L(p)$ and this implies $(x+t) \sim x$ and $(x+t) \in G(p)$.

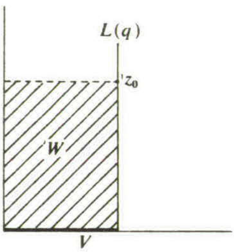


fig. 5.7.14

Let

$$V = \{z \in X \mid qz = 1 \wedge [z^i = 0 \Leftrightarrow q^i = 0]\}$$

$V \subset M(q)$ is closed and bounded. Choose ϵ such that

$$\epsilon > \max_{y \in V} d(x, y).$$

By axiom $C6$, there exists λ such that

$$d(x, z) < \epsilon \Rightarrow z + \lambda t \sim z + (\lambda + 1)t$$

Let

$$W = \{z \mid z \leq y + \lambda t \wedge y \in V\} \subset M(q)$$

Since W is closed and bounded, $u(x)$ attains a maximum in W in some point z_0. If $v \in M(q)$, then $z \in W$ and $\mu > 0$ exist, such that $z + \mu t \gtrsim v$ and $z \sim z + \mu t$, hence $z_0 \gtrsim z \sim z + \mu t \gtrsim v$, and $v \notin C(u(z_0))$.

This implies
$z_0 \in M(q) \cap C(u(z_0))$ and $M(q) \cap \text{Int } C(u(z_0)) = \varnothing$, hence $L(q)$ touches $C(u_0)$ in z_0, and $z_0 \in G(q)$.

As a direct consequence of this theorem we prove the convexity of P':

Theorem 5.7.15

$$p_1, p_2 \in P' \wedge \lambda \in [0,1] \Rightarrow \lambda p_1 + (1-\lambda) p_2 \in P'$$

Proof

Let $q = \lambda p_1 + (1-\lambda) p_2$. Now we have

$$q^i = 0 \Rightarrow (p_1{}^i = 0 \wedge p_2{}^i = 0)$$

Since $p_1 \in P'$, by theorem 5.7.14 also $q \in P'$.

There can exist $p \in P$ for which $L(p)$ is not a support of any preference set, because the utility function does not attain a maximum in $M(p)$. By theorem 5.7.12 this is only possible if $p \not> 0$. Now $M(p)$ is not bounded. In this case, $M(p)$ either intersects every $C(u)$, (e.g. for $p = 0$) or $L(p)$ is an asymptotic support for some $C(u)$. In the last case, $p \in C^*(u)$. This is the case treated in (c) above (see fig. 5.7.7).
If $\max\limits_{x \in M(p)} u(x)$ does not exist, $M(p)$ does not contain a greatest element, hence

$$H(M(p)) = \varnothing$$

and by axiom $C10$ this implies

$$M(p) \notin \mathscr{P} \text{ and } p \notin P'$$

Now $M(p)$ is not a choice set, since a choice set must contain a best element. Choice situations represented by $M(p)$ are considered as impossible. This means that a price vector with 0-component is impossible, if at that price of some commodity an unlimited quantity would be demanded. This is evident since unlimited quantities cannot be delivered.
(This is a more precise interpretation of axiom $C10$, not a conclusion.)
Summarising, we can state

$$x \in G(p) \Leftrightarrow [\exists u\colon x \in M(p) \cap C(u) \wedge M(p) \cap \text{Int } C(u) = \varnothing] \Leftrightarrow u(x) = \max_{y \in M(p)} u(y) \quad (5.7.16)$$

In principle it would not be impossible that $L(p)$ would be a support of some preference set $C(u_1)$, while it would be an asymptote of another set $C(u_2)$ so that $p \in \text{Bnd } C^*(u_1)$ and $p \in \text{Bnd } C^*(u_2)$. This case however, is ruled out by axiom $C6$. (It is *not* excluded that $L(p)$ is an *asymptotic* support of *both* $C(u_1)$ and $C(u_2)$.)

Theorem 5.7.17

$$p \in \text{Bnd}\, C^*(u_0) \wedge M(p) \cap C(u_0) \neq \varnothing \Rightarrow$$
$$(u > u_0 \Rightarrow \exists \mu < 1 \colon L(\mu p) \cap C(u) = \varnothing)$$

Proof

Let $x \in C(u_0) \cap M(p)$.
Choose t such that $t^i = 1$ if $p^i = 0$ and $t^i = 0$ if $p^i > 0$.
Now $px = p(x+t)$, hence $x+t \in L(p)$. We have $x+t \gtrsim x$, since $x \geq x+t$ and $x \gtrsim x+t$, since $x+t \notin \text{Int } C(u_0)$, hence $x \sim x+t$.
Choose φ such that $M(\varphi p) \cap C(u) \neq \varnothing$.

$$V = \{y \in X \mid \varphi p y \leqslant 1 \wedge (y^i = 0 \Leftrightarrow p^i = 0)\}$$

V is closed and bounded.
Choose ϵ such that $\epsilon > \max_{y \in V} d(x, y)$
By axiom $C6$, λ exists such that

$$d(x, z) < \epsilon \Rightarrow z + \lambda t \sim z + (\lambda + 1)t$$

Let $W = \{z \in X \mid \exists v \quad V \colon z \leq v + \lambda t\}$.
W is closed, bounded and convex and hence $T = W \cap C(u)$ is also closed, bounded and convex. $T \subset \text{Int } C(u_0)$, hence μ exists, such that

$$1 < \min_{z \in T} pz = \frac{1}{\mu} = pz_0$$

So $L(\mu p)$ must be a support of T.
We have

$$L(\mu p) \cap \text{Int}\, C(u) = \varnothing,$$

for, if $y \in L(\mu p)$, then $v \in V$ and $\tau \geqslant \lambda$ exist, such that $v + \tau t \geqq y$ and $z_0 \gtrsim v + \lambda t \sim v + \tau t \gtrsim y$.

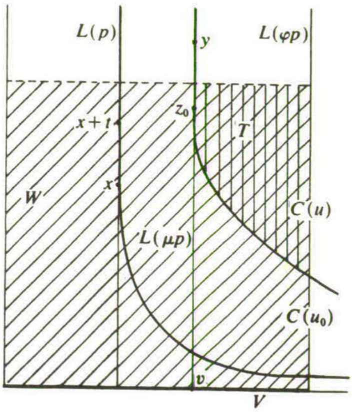


fig. 5.7.18

From theorem 4.4.22, we know that the dual of the dual of a closed c.u.p. set is identical to the original set. Hence

$$C(u) = C^{**}(u) = \{x \in X \mid M(x) \cap \operatorname{Int} C^*(u) \neq \varnothing\}$$

Since $M(x)$ contains all prices at which x can be bought, the preference set $C(u)$ can also be considered as the set of all commodity bundles, which cannot be bought at prices from the set $\operatorname{Int} C^*(u)$.

(5.7.16) states that a bundle x is demanded at a price p, if and only if $L(p)$ supports the preference set in x. By theorem 4.4.25 (see fig. 4.4.26)

$$L(p) \text{ supports } C(u) \text{ in } x \Leftrightarrow L(x) \text{ supports } C^*(u) \text{ in } p \qquad (5.7.19)$$

Applying this statement to (5.7.16), it follows

$$x \in G(p) \Leftrightarrow \exists u \in U: [p \in M(x) \cap C^*(u) \wedge M(x) \cap \operatorname{Int} C^*(u) = \varnothing] \qquad (5.7.20)$$

and taking into account definition 5.2.1,

$$x \in X' \Leftrightarrow \exists p \in P': L(p) \text{ supports } C(u(x)) \qquad (5.7.21)$$

and also

$$x \in X' \Leftrightarrow L(x) \text{ supports } C^*(u(x)) \qquad (5.7.22)$$

Now it follows (see theorem 5.7.12)

Theorem 5.7.23

$$x > 0 \Rightarrow x \in X'$$

Proof

Let $x > 0$ and $x \in \text{Bnd}\, C(u) = \text{Bnd}\, C^{**}(u)$, hence

$L(x) \cap \text{Int}\, C^*(u) = \varnothing$

Choose $\lambda < 1$, and let $D = L(\lambda x) \cap C^*(u) \neq \varnothing$. This intersection is closed and bounded and therefore the continuous function px attains a minimum in this set.

$$\min_{p \in D} px = p_0 x = \alpha$$

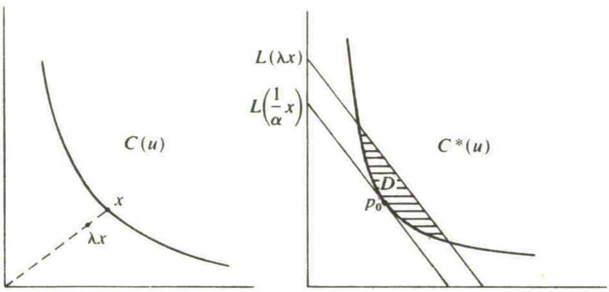


fig. 5.7.24

Hence $(1/\alpha)x \in \text{Bnd}\, C^{**}(u)$ and since $x \in \text{Bnd}\, C^{**}(u)$, we have $\alpha = 1$ so that $L(x)$ supports $C^*(u)$.

If $x \in \text{Int}\, C(u)$ and $u = u(x)$, then $x \in X'$, since there cannot exist $p \in P'$ such that $L(p)$ supports $C(u)$ in x.
This implies:

$$\text{Int}\, C(u) \cap X' = \{x \in X' \mid u(x) > u\} \tag{5.7.25}$$
$$\text{Bnd}\, C(u) \cap X' = \{x \in X' \mid u(x) = u\} \tag{5.7.26}$$

5.8 PREORDERING ON THE PRICE SPACE AND THE DUAL UTILITY FUNCTION

In 3.5 we introduced a preordering on the set of choice sets. In 5.5 we showed that we could replace this relation by a binary relation on the price space, where

$$p \gtrsim^* q \Leftrightarrow \exists x, y\colon x \in G(p) \wedge y \in G(q) \wedge x \gtrsim y$$

This relation is only defined on P'. It states that one price is at least as favourable as another, if and only if the bundle demanded at the first price is at least as good as the bundle demanded at the second price. Since the consumer preference model is a special case of the preference model, the relation $\gtrsim^{*}$ must be transitive and complete (theorem 3.5.3). Now the duals can also be interpreted by means of this preordering. A dual set consists of all prices that are *no more favourable* than a boundary point of this set. Hence its complement contains all prices that are more favourable than a boundary point.

Theorem 5.8.1

$$p \in \mathrm{Bnd}\, C^{*}(u) \cap P' \Rightarrow C^{*}(u) \cap P' = \{q | p \gtrsim^{*} q\}$$

Proof

If $p \in P'$, $x \in G(p)$ exists and, since $p \in \mathrm{Bnd}\ C^{*}(u)$, we have $x \in \mathrm{Bnd}\, C(u)$.

If $q \in C^{*}(u) \cap P'$, then $y \in G(q)$ exists and we have $M(q) \cap \mathrm{Int}\, C(u) = \varnothing$, hence $y \notin C(u)$ and this implies $x \gtrsim y$ and therefore $p \gtrsim^{*} q$.

If conversely $p \gtrsim^{*} q$, then $x \gtrsim y$, so that $y \in \mathrm{Int}\ C(u)$, hence $M(q) \cap \mathrm{Int}\, C(u) = \varnothing$ and therefore $q \in C^{*}(u)$.

Because $\gtrsim^{*}$ is complete on P', this theorem implies directly

$$p \in \mathrm{Bnd}\, C^{*}(u) \cap P' \Rightarrow P' - C^{*}(u) = \{q \mid q \succ^{*} p\} \tag{5.8.2}$$

and

$$p, q \in \mathrm{Bnd}\, C^{*}(u) \cap P' \Rightarrow p \sim^{*} q. \tag{5.8.3}$$

The interpretations of the preference sets and of their duals are much similar. Therefore we present these interpretations simultaneously.

Let $x_0 \in G(p_0)$ and $u_0 = u(x_0)$.

$C(u_0)$ is the set of all commodity bundles which are at *least as good as* x_0, and thus have utilities equal to or higher than u_0.

$C^{*}(u_0)$ is the set of all prices at which only commodity bundles can be bought which are *not better than* x_0, and thus have utilities equal to or lower than u_0. Hence the prices of $C^{*}(u_0)$ are *no more favourable than* p_0.

Bnd $C(u_0)$ contains nearly[1] all bundles which are *equivalent* to x_0, so that their utilities are equal to u_0.

Bnd $C^*(u_0)$ contains all prices at which bundles can be bought that are *equivalent* to x_0, but no better ones. Hence these prices are *equally favourable* as p_0.

Int $C(u_0)$ contains all commodity bundles that are *better* than x_0.

Int $C^*(u_0)$ contains nearly[2] all prices at which only commodity bundles can be bought that are *worse* than x_0. Hence these prices are *less favourable* than p_0.

$X - C(u_0)$ consists of all commodity bundles which are *worse* than x_0.

$P - C^*(u_0)$ consists of all prices at which bundles can be bought that are *better* than x_0. Hence these prices are *more favourable* than x_0.

1. Some bundles equivalent to x_0 may be in Int $C(u_0)$, but then $x \not> 0$ and $x \notin X'$ (see (5.7.26)).

2. Prices at which no bundles equivalent to x_0 can be obtained, may be in Bnd $C^*(u_0)$, but then $p \not> 0$ and $p \notin P'$ (see theorem 5.8.1).

The binary relation $\gtrsim^*$ on P' has roughly the same properties[14] as $\gtrsim$ on X. All properties of $\gtrsim$ established by the axioms of the present model also hold for $\gtrsim^*$, with the exception of $C6$. The properties stated in the next theorem correspond to axioms $C3$, $C4$, $C5$, $C7$ and $C8$ respectively.

Theorem 5.8.4

For $p, q, r \in P'$, we have

a. (transitivity)

$$p \gtrsim^* q \wedge q \gtrsim^* r \Rightarrow p \gtrsim^* r$$

b. (completeness)

$$p \gtrsim^* q \vee q \gtrsim^* p$$

14. Starting with a similar set of axioms with respect to $\gtrsim$, Milleron derived the properties (a), (b), (d) and (e) of theorem 5.8.4 in a similar way. See MILLERON.

c. (monotonicity)

$$q \geqq p \Rightarrow p \gtrsim^* q$$
$$q > p \Rightarrow p \succ q$$

d. (continuity)

$$p \gtrsim^* r \wedge r \gtrsim^* q \Rightarrow \exists\lambda: \lambda p + (1-\lambda)q \sim^* r$$

e. (convexity)

$\forall p_0: \{p \in P' \mid p_0 \gtrsim^* p\}$ is convex.

Proof

(a, b) have been proved in theorem 3.5.3.

c. Let $x \in G(p)$, hence $p \in \text{Bnd}\, C^*(u(x))$ and by theorem 5.8.1,

$P' \cap C^*(u(x)) = \{q \in P' \mid p \gtrsim^* q\}$.

Since $C^*(u(x))$ is a c.u.p. set, it follows

$q \geqq p \Rightarrow q \in C^*(u(x))$, hence $p \gtrsim^* q$

If $q > p$, then $q \in \text{Int}\, C^*(u(x))$, hence $p \succ^* q$.

d. If $z \in G(r)$ we have $r \in \text{Bnd}\, C^*(u(z))$, $q \in C^*(u(z))$ and $p \in P' - \text{Int}\, C^*(u(z))$ and by applying theorem 4.2.23 (see proof of theorem 5.3.7),

$\text{Bnd}\, C^*(u(z)) \cap \{s \in P' \mid \exists\lambda \in [0, 1]: s = \lambda p + (1-\lambda)q\} \neq \varnothing$.

e. If $x_0 \in G(p_0)$, then $p_0 \in \text{Bnd}\ C^*(u(x_0))$ and the set $P' \cap C^*(u(x_0)) = \{p \in P' \mid p_0 \gtrsim^* p\}$ is convex.

Because the set P' is completely preordered by $\gtrsim^*$ and since for this relation the continuity condition is fulfilled, it is possible to construct an order-preserving function on P'; hence there exists a mapping v: $P' \to R$, such that

$$v(p) \geqslant v(q) \Leftrightarrow p \gtrsim^* q$$

This construction can be performed in the same way as it was done in theorem 5.4.1 for u: $X \to R$. However, the function can also be obtained from $u(x)$ by associating with $p \in P'$, the utility index of the commodity bundle that is demanded at p.

Definition 5.8.5

The mapping $v: P' \to R$ is called dual utility function, if

$$v(p) = u \Leftrightarrow p \in \text{Bnd}\, C^*(u) \cap P'$$

Clearly, we now have v: $P' \to U$. The dual utility function (also called indirect utility function) is order-preserving: if $p \in$ Bnd $C^*(u_1)$ and $q \in$ Bnd $C^*(u_2)$, while $u_1 \geqslant u_2$, then $p \gtrsim^* q$. Since every $p \in P'$ is for some u in the boundary of $C^*(u) \cap P'$, the function v is defined on P'. Taking into account (5.7.20), we also have

$$v(p) = u \Leftrightarrow x \in G(p) \wedge u(x) = u, \tag{5.8.6}$$

and applying theorem 5.8.1, it follows:

$$\begin{aligned} &C^*(u) \cap P' = \{p \mid v(p) \leqslant u\} \\ &\text{Int}\, C^*(u) \cap P' = \{p \mid v(p) < u\} \\ &\text{Bnd}\, C^*(u) \cap P' = \{p \mid v(p) = u\} \end{aligned} \tag{5.8.7}$$

The dual utility function has properties similar to those of the utility function (see theorems 5.4.1, 5.4.5 and 5.4.6).

Theorem 5.8.8

The function v: $P' \to R$ is

a. continuous
b. non *increasing*: $p \geqq q \Rightarrow v(p) \leqslant v(q)$
c. quasi-*convex*:

$$v(p) = v(q) \Rightarrow \lambda v(p) + (1-\lambda) v(q) \geqslant v(\lambda p + (1-\lambda) q)$$

Proof

The proofs are similar to those of the corresponding properties of $u(x)$, now applying the conditions of theorem 5.8.4, instead of the axioms.

5.9 DEMAND FUNCTIONS, PRICE FUNCTIONS AND REVEALED PREFERENCE RELATIONS

By (5.7.16), we have

$$G(p) = \{x \in X' \mid u(x) = \max_{y \in M(p)} u(y)\}$$

and the set $G(p)$ consists of the points where $L(p)$ supports a preference set. In 5.8 we have shown, that $v(p) = u(x)$ if $x \in G(p)$. It can also be stated:

Theorem 5.9.1

$$x \in G(p) \Leftrightarrow u(x) = v(p) \wedge px = 1$$

Proof

$\Rightarrow$ follows directly from (5.8.6).
$\Leftarrow$ we have $C(u(x)) \cap L(p) \neq \emptyset$ and
$p \in C^*(u(x)) \cap L(x) \neq \emptyset$.
Suppose $\text{Int}\, C(u(x)) \cap L(p) \neq \emptyset$.
Then $p \notin C^*(u(x))$ and this is a contradiction.

The correspondence $G(p)$ is only defined for $p \in P'$, while $G(p) \subset X'$ always holds, hence G: $P' \rightarrow X'$, and this correspondence is *onto*. Remember that (see (2.3.3))

$$G(A) = \{x \mid \exists p: p \in A \wedge x \in G(p)\}$$

Clearly we have

$$G(P') = X'$$

while the demand function associates better commodity bundles with more favourable prices etc., and we can state:

Theorem 5.9.2

For $u \in U$

$$\text{Bnd}\, C(u) \cap X' = G(\text{Bnd}\, C^*(u) \cap P') \qquad \text{(a)}$$
$$\text{Int}\, C(u) \cap X' = G(P' - C^*(u)) \qquad \text{(b)}$$
$$X' - C(u) = G(\text{Int}\, C^*(u) \cap P') \qquad \text{(c)}$$

Proof

a. If $p \in \text{Bnd}\, C^*(u) \cap P'$, we have $G(p) \subset \text{Bnd}\, C(u) \cap X'$; if $x \in \text{Bnd}\, C(u) \cap X'$, then $L(x)$ supports $C^*(u)$.
b. If $p \in P'$, $v \in U$ exists, such that $p \in \text{Bnd}\, C^*(v)$ and if $p \in P' - \text{Int}\, C^*(u)$, then $v \geqslant u$, and applying (a) it follows
$G(p) \subset \text{Bnd}\, C(v) \cap X' \subset \text{Int}\, C(u) \cap X'$.
c. follows directly from (a) and (b), since $G(P') = X'$

For fixed values of p, the sets $G(p)$ have the following properties:[15]

15. It can also be shown that G is upper-semi-continuous for $p > 0$. See e.g. BERGE, pp. 117 and 122.

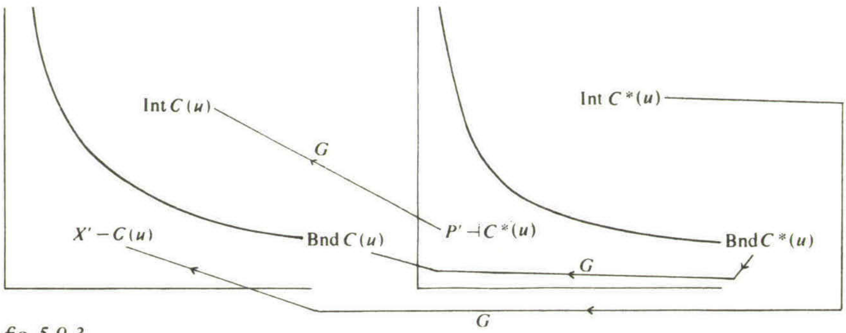


fig. 5.9.3

Theorem 5.9.4

$p \in P' \Rightarrow G(p)$ is convex and closed (a)
$p > 0 \Rightarrow G(p)$ is bounded (b)
$x \in G(p) \Rightarrow px = 1$ (c)

Proof

a. If $v(p) = u$, then $G(p) = C(u) \cap L(p)$ (see (5.7.16)); both sets are closed and convex, hence their intersection is also closed and convex.
b. For p strictly positive, $L(p)$ is also bounded, and therefore the intersection must also be bounded.
c. Since $x \in L(p)$, we have $px = 1$.

The revealed preference properties also hold for the demand function. It should be remembered that we have (see definition 3.4.1 and convention 5.5.3)

$$x\mathsf{R}y \Leftrightarrow [\exists p: x \in G(p) \wedge py \leqslant 1] \vee (x = y) \tag{5.9.5}$$

Hence x is revealed preferred to y, if x is demanded at a price, at which y also can be bought. Since the present model is a special case of the preference model, both the weak and the strong properties of revealed preference must hold. For the *weak* property we can now write (see theorem 3.4.6):

$$x \in G(p) \wedge y \notin G(p) \wedge py \leqslant 1 \Rightarrow \nexists q: qx \leqslant 1 \wedge y \in G(q) \tag{5.9.6}$$

it says: if x, and not y is demanded at some price p, and y could be bought at this price, then it is impossible that y is demanded at a price at which x could also be bought; hence the case sketched in fig. 5.9.7 is impossible. Clearly, $L(p)$ and $L(q)$ could never support the same c.u.p. set.

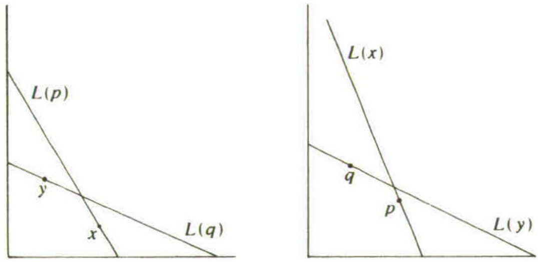


fig. 5.9.7

The strong property of revealed preference

$$x\bar{\mathsf{R}}y \Rightarrow y\not{\mathsf{P}}x.$$

is only relevant for $n > 2$ (in the two-commodity case it is implied by the weak property). Its meaning can be illustrated in the case of three commodities.

The following case can never occur:

$$\begin{array}{ll} x \in G(p) \text{ and } py \leqslant 1 & \left.\begin{array}{l}\text{hence } x\mathsf{R}y \\ \text{hence } y\mathsf{R}z\end{array}\right\} \text{hence } x\bar{\mathsf{R}}z \\ y \in G(q) \text{ and } pz \leqslant 1 & \\ z \in G(r) \text{ and } rx \leqslant 1 \text{ and } x \notin G(r) & \text{hence } z\mathsf{P}x \end{array}$$

This case is sketched[16] in figure 5.9.8. x, y and z are points on the edges of a cube. The plane $L(p)$ passes through $x \in G(p)$. On the cube are drawn the intersecting lines of $L(x)$ and the cube. The shaded area lies outside the budget set $M(p)$. The point y is in the budget set, hence $x\mathsf{R}y$ (actually $x\mathsf{P}y$). The dashed lines are the intersecting lines of $L(q)$ and the cube and y is in the intersection of $L(q)$ and the cube (since $y \in G(q)$). Since $z \in M(q)$, $y\mathsf{R}z$. Finally the dotted lines are the intersection of $L(r)$ and the cube, while $z \in G(r)$. Since $x \in M(r)$ and $x \notin G(r)$, we have $z\mathsf{P}x$.

This case is excluded by the strong property, however there is no contradiction with the weak property.

From the correspondence $G: P' \to X'$, can be derived an inverse correspondence $F: X' \to P'$. This correspondence associates with every commodity bundle the prices, at which it is demanded. It will be called a *price function*.

16. Similar figures were given by HOUTHAKKER (1950), p. 160 and SAMUELSON (1950), p. 368.

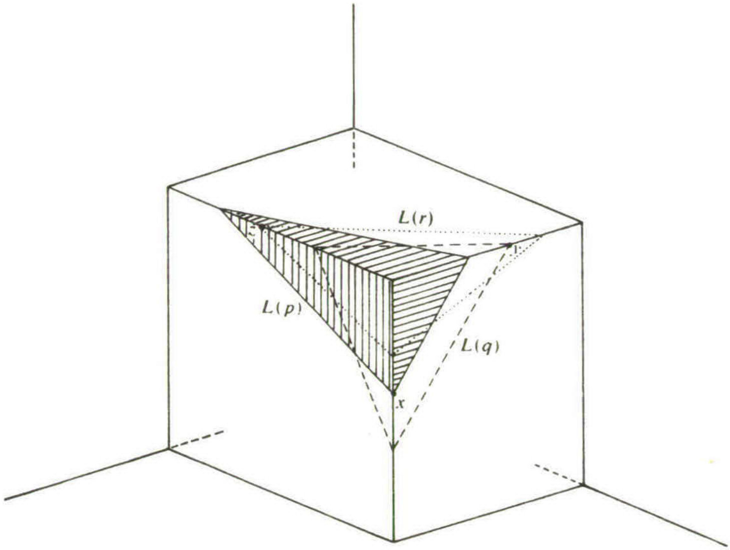


fig. 5.9.8

Definition 5.9.9

For $x \in X'$

$$F(x) = \{p \in P' \mid x \in G(p)\}$$

This correspondence is *onto*. $F(x)$ consists of all prices in which the hyperplane $L(x)$ supports a dual set $C^*(u)$, and therefore $F(x)$ contains the *least favourable prices* at which x can be bought.

Theorem 5.9.10

$$\begin{aligned} F(x) &= \{p \in P' \mid \exists u\colon L(x) \text{ supports } C^*(u) \text{ in } p\} \\ &= \{p \in P' \mid v(p) = \min_{q \in M(x)} v(q)\} \end{aligned}$$

Proof

If $p \in F(x)$, by definition we have $x \in G(p)$. Hence $L(p)$ supports $C(u(x))$ in x. By (5.7.19), this implies that $L(x)$ supports $C^*(u(x))$ in p. Hence the dual utility function v attains its minimum on $M(x)$ in p.

The price function has the same properties as the demand function and these can be derived in a similar way from the dual sets.

Theorem 5.9.11

For $u > 0$

$$F(\operatorname{Bnd} C(u) \cap X') = \operatorname{Bnd} C^*(u) \cap P' \qquad \text{(a)}$$
$$F(\operatorname{Int} C(u) \cap X') = P' - C^*(u) \qquad \text{(b)}$$
$$F(X' - C(u)) = \operatorname{Int} C^*(u) \cap P' \qquad \text{(c)}$$

Theorem 5.9.12

$$x \in X' \Rightarrow F(x) \text{ is convex and closed} \qquad \text{(a)}$$
$$p > 0 \Rightarrow F(x) \text{ is bounded} \qquad \text{(b)}$$
$$p \in G(x) \Rightarrow px = 1 \qquad \text{(c)}$$

For the direct revealed favourability relation holds:

$$p\mathsf{R}^{\star}q \Leftrightarrow G(q) \cap M(p) \neq \varnothing \qquad (5.9.13)$$

and hence

$$p\mathsf{R}^{\star}q \Leftrightarrow \exists x\colon q \in F(x) \wedge p \in M(x) \qquad (5.9.14)$$

The weak and the strong properties of revealed favourability for $\mathsf{R}^{\star}$ and $\bar{\mathsf{R}}^{\star}$ require:

$$p \in F(x) \wedge q \in M(x) \wedge q \notin F(x) \Rightarrow \nexists y\colon p \in M(y) \wedge q \in F(y) \qquad (5.9.15)$$

and

$$p\bar{\mathsf{R}}^{\star}q \Rightarrow q\not{\mathsf{P}}p \qquad (5.9.16)$$

Their meaning in dual space is the same as that of the corresponding properties in commodity space (see the right hand part of fig. 5.9.7; a figure like 5.9.8 can also be constructed in P).

Both the commodity space and the price space are subsets of n-dimensional euclidean space, and hence a distance can be defined. Therefore, it is possible to introduce a third revealed preference concept, henceforward called *extended revealed preference* (favourability). By this relation, a point x is also considered to be revealed preferred to y if either in every neighbourhood of x lies a point that is revealed preferred to y (by $\bar{\mathsf{R}}$) or in every neighbourhood of y lies a point to which x is revealed preferred.

Definition 5.9.17

$$x\bar{\bar{\mathsf{R}}}y \Leftrightarrow [\forall \epsilon, \exists z\colon (z \in B_\epsilon(y) \wedge x\bar{\mathsf{R}}z) \vee (z \in B_\epsilon(x) \wedge z\bar{\mathsf{R}}y)]$$
$$p\bar{\bar{\mathsf{R}}}^{\star}q \Leftrightarrow [\forall \epsilon, \exists r\colon (r \in B_\epsilon(q) \wedge p\bar{\mathsf{R}}^{\star}q) \vee (r \in B_\epsilon(p) \wedge r\bar{\mathsf{R}}^{\star}q]$$

Clearly we have

$$x\bar{\mathsf{R}}y \Rightarrow x\bar{\bar{\mathsf{R}}}y \tag{5.9.18}$$

Between the revealed preference relations $\bar{\mathsf{R}}$ and $\bar{\bar{\mathsf{R}}}$ can never arise a contradiction:

Theorem 5.9.19

$$x\bar{\bar{\mathsf{R}}}y \Rightarrow x \gtrsim y$$

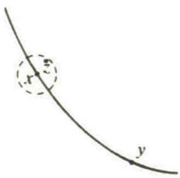


fig. 5.9.20

Proof

If $z\bar{\mathsf{R}}y$, we have $z \gtrsim y$ and hence $z \in C(u(y))$. Now let

$\forall \epsilon, \exists z: z \in B_\epsilon(x) \wedge z\bar{\mathsf{R}}y$.

This implies

$\forall \epsilon: B_\epsilon(x) \cap C(u(y)) \neq \emptyset$.

Since $C(u(y))$ is closed, by theorem 4.2.14, $x \in C(u(y))$, hence $x \gtrsim y$.

This directly implies

$$x\bar{\bar{\mathsf{R}}}y \Rightarrow y \not\bar{\mathsf{R}} x \tag{5.9.21}$$

6. A demand function model

6.1 INTRODUCTION

In the preceding chapter was treated a consumer choice model based on a preference relation, which was a special case of model P of chapter III. The demand function model is a special case of model K of chapter III. The primitive concepts of this model are identical to those of model C, and the axioms particularly refer to the demand function.
The axioms are chosen in such a way, that the axioms of model D can be derived as theorems. This idea is very old and was formulated originally:[1] What conditions must a single valued, differentiable demand function satisfy, in order that a utility function can be derived? The axioms of the present model are sufficient to guarantee that the binary relation $\gtrsim$ is a complete preordering, and also imply that the other axioms are true.
As far as we know the problem has only been treated[2] starting with single valued demand functions. The model of this chapter is more general since the demand function is a correspondence. Besides this, we only apply topological arguments and do not use differential equations.
In section 6.3 the axioms are presented, in 6.4 preference sets in the commodity space and their duals in the price space are reconstructed, with the help of revealed preference, from the demand function. In the final section the axioms of the consumer preference model are derived as theorems.

1. See SAMUELSON (1950) for a historical survey, and also GEORGESCU ROEGEN, Part II, 1 and 4.
2. VILLE, HOUTHAKKER (1950), SAMUELSON (1950), UZAWA (1960).

6.2 PRIMITIVE CONCEPTS

The primitive concepts of the demand function model are identical to those of the consumer choice model (section 5.2):

$X \subset R_+{}^n$	(choice space)
$\tilde{P} \subset R_+{}^{n+1}$	(price income space)
$\gtrsim$	(preference relation)
$\mathscr{P}$	(set of choice sets)
$K: \mathscr{P} \to X$	(choice function)

Before we introduce the axioms of this model, we repeat the definitions of some concepts, derived from the preliminary ones, because most axioms are expressed in terms of the derived concepts.
Price space (relative prices), (definition 5.2.2):

$$P = \left\{ p \in R_+{}^n \mid \exists \tilde{p} \in \tilde{P}: p^i = \frac{\tilde{p}^i}{\tilde{m}} \text{ for } i \in \{1, 2, \ldots, n\} \right\}$$

The set of commodity bundles that are eligible from at least one choice set (definition 5.2.1):

$$X' = \{x \in X \mid \exists M \in \mathscr{P}: x \in K(M)\}$$

The set of prices that correspond to a choice set (definition 5.2.3):

$$P' = \{p \in P \mid M(p) \in \mathscr{P}\}.$$

The demand function $G: P' \to X'$ (definition 5.5.1):

$$\forall p \in P': G(p) = K(M(p))$$

The price function, inverse of the correspondence G, F: $X' \to P'$ (definition 5.9.9):

$$\forall x \in X': F(x) = \{p \in P' \mid x \in G(p)\}$$

Finally, all revealed preference and favourability concepts are used (R, R^k, $\bar{\mathsf{R}}$, $\bar{\bar{\mathsf{R}}}$ etc.).

6.3 THE AXIOMS OF THE DEMAND FUNCTION MODEL

Most axioms of this model appear either as axioms or as theorems in the consumer preference model. Only $D1$ (b) and (c), $D9$ and $D10$ cannot be derived from model C. $D1$ (b) and (c) are of minor importance, $D9$ and $D10$ are more fundamental.

$D1$ (Range of the demand function)

$$x \in X \wedge x > 0 \Rightarrow x \in X' \tag{a}$$
X' is convex (b)
$$x \in X' \wedge \lambda > 0 \Rightarrow \lambda x \in X' \tag{c}$$

$D2$ (Domain of the demand function)

$$p \in P \text{ and } p > 0 \Rightarrow p \in P' \tag{a}$$
P' is convex (b)
$$p \in P' \wedge \lambda > 0 \Rightarrow \lambda p \in P' \tag{c}$$

$D3$ (Extent of the set of choice sets)

$$\mathscr{P} = \{M \subset X' \mid \exists p \in P' : M = M(p)\}$$

$D4$ (Closedness)

$$\forall p \in P' : G(p) \text{ is closed}$$

$D5$ (Budget equation)

$$x \in G(p) \Rightarrow px = 1$$

$D6$ (Weak axiom of revealed preference)

$$x \in G(p) \wedge y \notin G(p) \wedge py \leq 1 \Rightarrow \nexists q \in P' : y \in G(q) \wedge qx \leq 1$$

$D7$ (Strong axiom of revealed preference)

$$p\bar{R}q \Rightarrow q \not{R} p$$

$D8$ (Weak satiation axiom)

$$x \in G(p) \wedge x + t \in G(p) \wedge t \geqq 0 \Rightarrow \forall \epsilon > 0, \exists \lambda > 0 : [y \in B_\epsilon(x) \Rightarrow \exists q : y + \lambda t \in G(q) \wedge y + (\lambda + 1)t \in G(q)]$$

$D9$ (Generalised Lipschitz condition for G)

$$\exists \varphi > 0, \forall p \in P', \forall \lambda > 0 : \left[x \in G(p) \Rightarrow \exists y \in X' : y \in G(\lambda p) \wedge \left| y - \frac{1}{\lambda} x \right| < \varphi \left| \frac{1-\lambda}{\lambda} \right| \frac{1}{|p|} \right]$$

$D10$ (Generalised Lipschitz condition for F)

$$\exists \Psi > 0, \forall x \in X', \forall \lambda > 0 : \left[p \in F(x) \Rightarrow \exists q \in P' : q \in F(\lambda x) \wedge \left| q - \frac{1}{\lambda} p \right| < \Psi \left| \frac{1-\lambda}{\lambda} \right| \frac{1}{|x|} \right]$$

$D11$ (Transition axiom)

$$x\bar{\bar{R}}y \Rightarrow x \gtrsim y \qquad \text{(a)}$$
$$x\bar{P}y \Rightarrow x \succ y \qquad \text{(b)}$$

These axioms will now be considered consecutively:

Axiom D1

$$x \in X \wedge x > 0 \Rightarrow x \in X' \qquad \text{(a)}$$

Every strictly positive commodity bundle is demanded at some price. This part of the axiom is identical to theorem 5.7.26.

$$X' \text{ is convex} \qquad \text{(b)}$$
$$x \in X' \wedge \lambda > 0 \Rightarrow \lambda x \in X' \qquad \text{(c)}$$

(b) and (c) imply that X' is a c.u.p. set whereas (c) also requires that every point on a half-line from the origin (except the origin itself) is in X', if one point of this line is in X'. Both (b) and (c) only refer to commodity bundles with 0-component, since they are implied by (a) for positive bundles. Hence, they are of minor importance and only serve to avoid certain complications that might arise otherwise. They are not implied by model C.

Axiom D2

$$p \in P \wedge p > 0 \Rightarrow p \in P' \qquad \text{(a)}$$
$$P' \text{ is convex} \qquad \text{(b)}$$
$$p \in P' \wedge \lambda > 0 \Rightarrow \lambda p \in P' \qquad \text{(c)}$$

P' must satisfy the same conditions as X'. (a) and (b) are identical to theorems 5.7.12 and 5.7.15, whereas (c) is implied by 5.7.13.

Axiom D3

$$\mathscr{P} = \{M \subset X' \mid \exists p \in P' : M = M(p)\}$$

The choice function is only defined for budget sets, that are associated with prices of P', hence the choice function coincides with the demand function. This axiom corresponds to axiom $C10$.

Axiom D4

$$\forall p \in P' : G(p) \text{ is closed.}$$

This is a part of theorem 5.9.4.

Axiom D5

$$x \in G(p) \Rightarrow px = 1$$

This is implied by theorem 5.9.1, and requires that income is entirely spent.

Axioms $D6$ and $D7$ are the weak and the strong axioms[3] of revealed preference. They are implied by model C (see section 5.5 and theorems 3.4.5 and 3.4.15), and also appeared as theorems in model P.

Axioms $D1$, $D5$ and $D6$ directly imply that $G(p)$ is convex, as was also stated in theorem 5.9.4(a).

Theorem 6.3.1.

$$\forall p \in P': G(p) \text{ is convex}$$

Proof

Let $x_1, x_2 \in G(p), x = \lambda x_1 + (1-\lambda)x_2$ for $\lambda \in [0, 1]$.
We have $px_1 = px_2 = 1 = px$.
Suppose $x \notin G(p)$. Now by $D1$(b), q exists such that $x \in G(q)$, hence $qx = 1$. Now we must have

$$qx_1 \geqslant 1 \vee qx_2 \geqslant 1$$

and this is impossible by axiom $D6$.

Axiom D8

$$x \in G(p) \wedge x+t \in G(p) \wedge t \geqq 0 \Rightarrow$$
$$\forall \epsilon > 0, \exists \lambda > 0: [y \in B_\epsilon(x) \Rightarrow$$
$$\exists q: y+\lambda t \in G(q) \wedge y+(\lambda+1)t \in G(q)]$$

This axiom only refers to prices that are not strictly positive, since for $p > 0$ it is impossible that $x \in G(p)$ and $x+t \in G(p)$. ($px = p(x+t)$ implies $pt = 0$, hence $t \ngtr 0$.)
This axiom is the demand function version of axiom $C6$ and in the consumer preference model it is directly implied by $C6$ (see theorem 5.7.17).

3. Uzawa (1959) and Moeseke have shown that in models with single valued demand function, the strong axiom may be replaced by another axiom.

Axiom D9

$$\exists\varphi > 0, \forall p \in P', \forall\lambda > 0: \left[x \in G(p) \Rightarrow \exists y \in X': y \in G(\lambda p) \wedge \left| y - \frac{1}{\lambda} x \right| < \varphi \left| \frac{1-\lambda}{\lambda} \right| \frac{1}{|p|} \right]$$

This axiom can be considered as a continuity condition. It is not implied by model C. (It differs from upper-semi-continuity.) It is a version of the so-called Lipschitz condition, which is usually assumed in models with a single valued demand function.[4] This axiom is used to prove the completeness of the preference relation.

Let the bundle x be demanded at a price p. Suppose that this price is increased or decreased proportionally, i.e. that all components of this price increase or decrease with the same percentages. Hence a new price λp results, with $\lambda > 0$. If at a proportional price increase (decrease) the demanded commodity bundle would decrease (increase) proportionally, then the bundle $(1/\lambda)x$ would be demanded at λp.

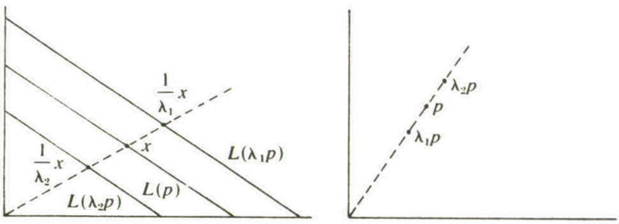


fig. 6.3.2

Now the axiom requires that a commodity bundle is demanded at λp, which does not differ too much from $(1/\lambda)x$. A point of $G(\lambda p)$ must exist that lies within a certain distance from $(1/\lambda)x$, and this distance depends on

a. a constant φ

b. the distance $|(1-\lambda)/\lambda|(1/|p|)$ between the budget planes $L(p)$ and $L(\lambda p)$.

Hence a proportional change of the commodity bundle is considered as a normal reaction to a proportional price change. Deviations from this reaction must remain within certain boundaries. Thus, it is excluded that a small proportional price change causes abrupt changes in the composition of the commodity bundle. (At large proportional

4. See e.g. Houthakker (1950) and Uzawa (1960).

price changes, where λ differs considerably from 1, the axiom is not a serious restriction.)

In the two-dimensional case (see fig. 6.3.3, where $\lambda < 1$), the plane $L(\lambda p)$ contains two points y_1 and y_2, such that

$$\left|y_1 - \frac{1}{\lambda}x\right| = \left|y_2 - \frac{1}{\lambda}x\right| = \varphi\left|\frac{1-\lambda}{\lambda}x\right|\frac{1}{|p|}$$

where $x \in G(p)$. The number

$$\left|\frac{1-\lambda}{\lambda}\right|\frac{1}{|p|} = |r_1 - r_2|$$

is the distance between the planes $L(p)$ and $L(\lambda p)$:

$$\frac{p'}{|p|^2} = r_1 \in L(p), \text{ since}^5\ pr_1 = p\frac{p'}{|p|^2} = 1$$

and

$$r_2 = \frac{1}{\lambda}\frac{p'}{|p|^2} \in L(\lambda p), \text{ since } \lambda pr_2 = 1$$

hence

$$|r_1 - r_2| = \left|1 - \frac{1}{\lambda}\right|\frac{|p|}{|p|^2} = \left|\frac{1-\lambda}{\lambda}\right|\frac{1}{|p|}$$

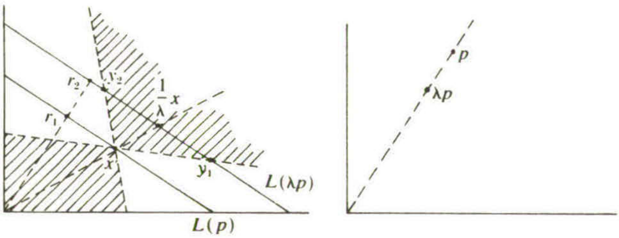


fig. 6.3.3

Now the axiom requires that $G(p)$ contains a point between y_1 and y_2. In general, we can state that for every $\lambda > 0$ the set $G(\lambda p)$ must intersect the shaded area.

The axiom implies a very interesting property in price space:
Let $x \in G(p)$, then the set

$$Q(p, x) = \left\{q \in P' \mid \exists\mu: |q - \mu p| \leqslant \frac{|p|}{\varphi} \wedge q \in L(x)\right\} \qquad (6.3.4)$$

5. Points of p are row vectors and points of X are column vectors, hence p' is a column vector and pp' is the product of a row and a column.

is a closed and bounded subset of the **plane** $L(x)$ that consists of all points of this plane, lying within a certain distance from the *half-line from the origin through p*.

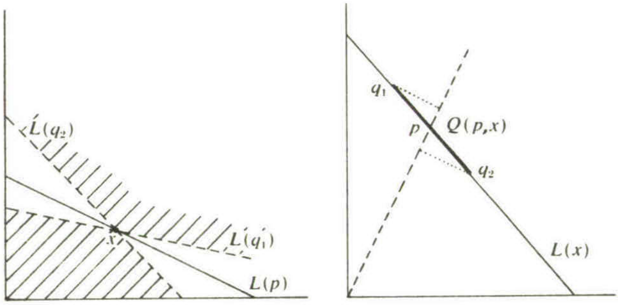


fig. 6.3.5

For each $\lambda > 0$, $G(\lambda p)$ must contain a point y, such that

$$L(y) \cap Q(p,x) = \emptyset. \tag{6.3.6}$$

for $\lambda > 1$ this means

$$M(y) \supset Q(p,x)$$

hence *all* prices of $Q(p, x)$ are directly revealed at least as favourable as λp:

$$q \in Q(p,x) \Rightarrow q\mathsf{R}^*\lambda p$$

Theorem 6.3.7

For $Q(p,x)$ as defined in (6.3.4):

$$\forall p \in P' : [x \in G(p) \wedge \lambda > 0 \wedge \lambda \neq 1 \Rightarrow$$
$$\exists y \in X' : y \in G(\lambda p) \wedge L(y) \cap Q(p,x) = \emptyset]$$

Proof

Let y be a point such that the condition of axiom $D9$ is satisfied and $q \in Q(p,x)$:

$$\left|y - \frac{1}{\lambda}x\right| < \varphi \left|\frac{1-\lambda}{\lambda}\right| \frac{1}{|p|}$$

Since

$$|q - \mu_0 p| = \min_{\mu} |q - \mu p| \leqslant \frac{|p|}{\varphi}$$

we have

$$-|q-\mu_0 p|\left|y-\frac{1}{\lambda}x\right| \leqslant (q-\mu_0 p)\left(y-\frac{1}{\lambda}x\right) \leqslant |q-\mu_0 p|\left|y-\frac{1}{\lambda}x\right|$$

hence

$$-\left|\frac{1-\lambda}{\lambda}\right| < (q-\mu_0 p)\left(y-\frac{1}{\lambda}x\right) < \left|\frac{1-\lambda}{\lambda}\right|$$

Since

$$\mu_0 p\frac{1}{\lambda}x = \mu_0\, py \text{ and } q\frac{1}{\lambda}x = \frac{1}{\lambda}$$

we have

$$-\left|\frac{1-\lambda}{\lambda}\right| < qy-\frac{1}{\lambda} < \left|\frac{1-\lambda}{\lambda}\right|$$

This implies

for $\lambda > 1$: $\dfrac{1-\lambda}{\lambda}+\dfrac{1}{\lambda} < qy < \dfrac{\lambda-1}{\lambda}+\dfrac{1}{\lambda}$ hence $qy < 1$

for $\lambda < 1$: $\dfrac{\lambda-1}{\lambda}+\dfrac{1}{\lambda} < qy < \dfrac{1-\lambda}{\lambda}+\dfrac{1}{\lambda}$ hence $qy > 1$

Therefore $qy \neq 1$ and hence $q \notin L(y)$.

The theorem can also be formulated

$$\exists\varphi > 0, \forall p \in P'\colon \Big(x \in G(p) \wedge \lambda > 0 \wedge \lambda \neq 1 \Rightarrow$$
$$\exists y \in X'\colon \Big[y \in G(\lambda p) \wedge q \in L(y) \cap L(x) \Rightarrow$$
$$\min_{\mu} |q-\mu p| > \frac{|p|}{\varphi}\Big]\Big) \qquad (6.3.8)$$

Axiom D10

The price function F, inverse of G, must have the same property as G, as was stated in axiom $D9$. Hence, a theorem, similar to theorem 6.3.7, can be deduced.

*Axiom D*11

It connects the demand function and the preference relation by means of revealed preference. The axiom corresponds to theorem 5.9.19 and the last part of theorem 3.4.18.

6.4 RECONSTRUCTION OF THE PREFERENCE SETS

In definition 4.4.3 were defined the correspondences $M: P \to X$ etc. In the rest of this chapter, these correspondences will be considered as correspondences of P' into X' (and of X' into P'), hence for $x \in X'$:

$$M(x) = \{p \in P' \mid px \leq 1\} \quad \text{for } x \in X'$$

Now $M(x)$ is the set of all prices at which the bundle x can be bought and for which the demand function is defined.
With the help of the demand function $G: P' \to X'$ we define a *composite correspondence* $GM: X' \to X'$, where

$$GM(x) = \{y \in X' \mid \exists p: p \in M(x) \wedge y \in G(p)\} \tag{6.4.1}$$

Thus with every $x \in X'$ are associated all commodity bundles that are demanded at prices, at which the bundle x can also be bought.

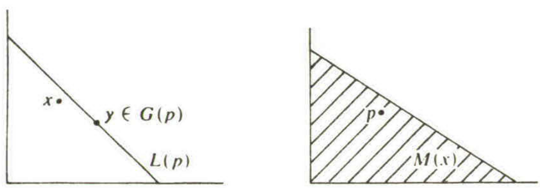


fig. 6.4.2

This implies, that $GM(x)$ contains all bundles that are directly revealed preferred to x, for if $y \in GM(x)$, p must exist such that $y \in G(p)$ and $x \in M(p)$, and by definition 3.4.1 (taking into account convention 5.5.3) this means $y\mathrm{R}x$.
Now with every set $GM(x)$, corresponds through the correspondence M, a set in P':

$$MGM(x) = \{p \in P' \mid \exists y: y \in GM(x) \wedge p \in M(y)\}$$

and by repeating this procedure a sequence of correspondences can be constructed:

$$MGM(x), GMGM(x), MGMGM(x), \ldots$$

where $MGM(x)$ is a subset of P', $GMGM(x)$ a subset of X' etc.
To simplify the notation we define ($N = \{1, 2, 3, \ldots\ldots\}$:

Definition 6.4.3

If $x \in X'$ and $k \in N$,

$$(GM)^{k+1}(x) = GM(GM)^k(x)$$

Thus we get two sequences of correspondences:

in X': $x, GM(x), (GM)^2(x), (GM)^3(x), \ldots.$

in P': $M(x), MGM(x), M(GM)^2(x), M(GM)^3(x), \ldots.$

We have already shown that $y\mathsf{R}x$ for $y \in GM(x)$. The following terms of the first sequence (the second will be considered later) have a similar meaning: the set $(GM)^k(x)$ consists of all commodity bundles that are *preferred* to x *in at most k steps*. Hence if $y \in (GM)^k(x)$, we have $y\mathsf{R}^kx$ (see definition 3.4.10). It follows that the union of all terms of the sequence contains the points that are *indirectly revealed preferred* to x (see definition 3.4.11).

Definition 6.4.4

If $x \in X'$,

$$\overline{GM}(x) = \{y \in X' \mid \exists k \in N: y \in (GM)^k(x)\} = \bigcup_{k \in N} (GM)^k(x).$$

Theorem 6.4.5

If $x \in X'$ and $k \in N$,

$$(GM)^k(x) = \{y \in X' \mid y\mathsf{R}^kx\} \quad \text{(a)}$$
$$\overline{GM}(x) = \{y \in X' \mid y\bar{\mathsf{R}}x\} \quad \text{(b)}$$

Proof

a. $k = 1$:

$$\begin{aligned} GM(x) &= \{y \in X' \mid \exists p{:}p \in M(x) \wedge y \in G(p)\} \\ &= \{y \in X' \mid \exists p{:}\, x \in M(p) \wedge y \in G(p)\} \\ &= \{y \in X' \mid y\mathsf{R}x\}. \end{aligned}$$

$k > 1$:

Assume that the theorem is true for k.

We prove that it is also true for $k+1$.

$$
\begin{aligned}
(GM)^{k+1}(x) &= \{y \in X' \mid \exists z, \exists p\colon y \in G(p) \wedge p \in M(z) \\
&\qquad\qquad\qquad\qquad \wedge z \in (GM)^k(x)\} \\
&= \{y \in X' \mid \exists z, \exists p\colon y \in G(p) \wedge z \in M(p) \wedge \\
&\qquad\qquad\qquad\qquad \wedge z \in (GM)^k(x)\} \\
&= \{y \in X' \mid \exists z\colon y\mathsf{R}z \wedge z\mathsf{R}^k x\} \\
&= \{y \in X' \mid y\mathsf{R}^{k+1}x\}
\end{aligned}
$$

b. $\overline{GM}(x) = \{y \in X' \mid \exists k \in N\colon y\mathsf{R}^k x\} = \{y \in X' \mid y\bar{\mathsf{R}}x\}$.

The sequence $(GM)^k(x)$ is increasing: as the number of steps increases, more preferences can be revealed. The sequence converges to $\overline{GM}(x)$, which appears to be a c.u.p. set.
We first prove some properties of the sets in the sequence in X'.

Theorem 6.4.6

If $x \in X'$ and $k \in N$,

$$x \in (GM)^k(x) \tag{a}$$
$$y \in (GM)^k(x) \Rightarrow GM(y) \subset (GM)^{k+1}(x) \tag{b}$$
$$(GM)^k(x) \subset (GM)^{k+l}(x) \quad (\text{for } l \in N) \tag{c}$$
$$\text{Conv}\,(GM)^k(x) \subset (GM)^{k+1}(x) \tag{d}$$
$$z \geqq y \wedge y \in (GM)^k(x) \Rightarrow z \in (GM)^{k+1}(x) \tag{e}$$

Proof

a. If $x \in G(p)$, we have $p \in M(x)$, hence $x \in GM(x)$.
b. $GM(y) \subset GM(GM)^k(x) = (GM)^{k+1}(x)$.
c. $x \in (GM)^l(x)$, hence $GM(x) \subset (GM)^{l+1}(x)$,
$(GM)^2(x) \subset (GM)^{l+2}(x)$, etc.
d. Let $y_1, y_2 \in (GM)^k(x)$ and $y = \mu y_1 + (1-\mu)y_2$ for $\mu \in [0, 1]$
For $y \in G(q)$, $qy = 1 = \mu q y_1 + (1-\mu)qy_2$ and this is possible if and only if (see fig. 6.4.7)

$$qy_1 \leqslant 1 \vee qy_2 \leqslant 1$$

Hence

$$q \in M(y_1) \vee q \in M(y_2)$$

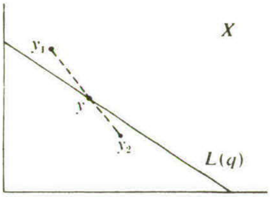


fig. 6.4.7

and also

$y \in GM(y_1) \vee y \in GM(y_2)$

and applying (b) it follows

$y \in (GM)^{k+1}(x).$

e. For $z \geqq y$ and $z \in G(q)$, we have $qz = 1 \geqslant qy$, hence $q \in M(y)$, and so $z \in GM(y) \subset (GM)^{k+1}(x)$.

This theorem entails

Theorem 6.4.8

If $x \in X'$,

$\overline{GM}(x)$ is a c.u.p. set (a)

$y \in \overline{GM}(x) \Rightarrow \overline{GM}(y) \subset \overline{GM}(x)$ (b)

Proof

a. $\overline{GM}(x)$ is *convex*: let $y_1, y_2 \in \overline{GM}(x)$, then $k \in N$ exists such that $y_1, y_2 \in (GM)^k(x)$ and if, $y = \mu y_1 + (1-\mu)y_2$ for $\mu \in [0, 1]$ we have, by theorem 6.4.6(d)

$y \in \text{Conv}\,(GM)^k(x) \subset (GM)^{k+1}(x) \subset \overline{GM}(x).$

$\overline{GM}(x)$ is unbounded above: $z \geqq y$ and $y \in \overline{GM}(x)$, therefore $k \in N$ exists, such that $y \in (GM)^k(x)$ and by theorem 6.4.6(c), this implies

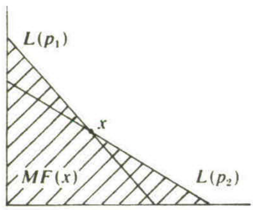
$z \in (GM)^{k+1}(x) \subset \overline{GM}(x).$

b. is directly implied by theorem 6.4.6(b).

Besides sets of points that are revealed preferred to some commodity bundle x, we can also construct sets of points to which x is preferred. Given the correspondences F: $X' \to P'$ and M: $P' \to X'$, we form the composite correspondence MF: $X' \to X'$. $F(x)$ contains prices at which x is demanded, and $MF(x)$ consists of all bundles that can also be bought at these prices.

$$MF(x) = \{y \in X' \mid \exists p: p \in F(x) \wedge y \in M(p)\} \quad \text{for } x \in X'.$$

For all points of this set $x\mathrm{R}y$ (since $x \in G(p)$ and $y \in M(p)$).



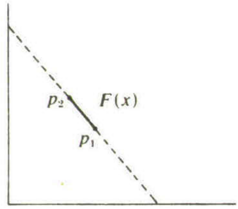


fig. 6.4.9

Another sequence of correspondences is constructed. We define

Definition 6.4.10

If $x \in X'$ and $k \in N$

$$(MF)^{k+1}(x) = MF(MF)^k$$

Now two sequences of correspondences exist:

in X': $x, MF(x), (MF)^2(x), (MF)^3(x), \dots.$
in P': $F(x), FMF(x), F(MF)^2(x), F(MF)^3(x), \dots.$

The first sequence consists of sets of points to which x is revealed preferred in k steps. $\overline{MF}$ is the union of all terms in this sequence.

Definition 6.4.11

If $x \in X'$

$$\overline{MF}(x) = \{y \in X' \mid \exists k \in N: y \in (MF)^k(x)\} = \bigcup_{k \in N} (MF)^k(x)$$

Theorem 6.4.12

If $x \in X'$ and $k \in N$,

$$(MF)^k(x) = \{y \in X' \mid x\mathrm{R}^k y\} \qquad \text{(a)}$$

$$\overline{MF}(x) = \{y \in X' \mid x\bar{\mathrm{R}}y\} \qquad \text{(b)}$$

Proof

The argument is similar to the proof of theorem 6.4.5.

The sets $(MF)^k(x)$ have similar properties as the sets $(GM)^k(x)$.

Theorem 6.4.13

If $x \in X'$ and $k \in N$,

$$x \in (MF)^k(x) \tag{a}$$

$$y \in (MF)^k(x) \Rightarrow MF(y) \subset (MF)^{k+1}(x) \tag{b}$$

$$(MF)^k(x) \subset (MF)^{k+l}(x) \qquad (\text{for } l \in N) \tag{c}$$

$$X' - (MF)^k(x) \text{ is convex.} \tag{d}$$

$$z \leqq y \wedge y \in (MF)^k(x) \Rightarrow z \in (MF)^k(x) \tag{e}$$

Proof

a, b, c See proof of theorem 6.4.6(a), (b), (c).
d. $y_1, y_2 \notin (MF)^k(x) = \{z \mid \exists p\colon p \in F(MF)^{k-1}(x) \wedge z \in M(p)\}$
hence if $p \in F(MF)^{k-1}(x)$, we have

$py_1 > 1$ and $py_2 > 1$

and for $\mu \in [0, 1]$, we have

$p(\mu y_1 + (1-\mu)y_2) > 1$

e. There exists $p \in F(MF)^{k-1}(x)$, such that $y \in M(p)$ and since $z \leqq y$, also $z \in M(p)$; hence $z \in (MF)^k(x)$.

The sets $X - (MF)^k(x)$ consist of the commodity bundles to which x is *not* revealed preferred in k steps. These sets appear to be c.u.p. (by (d) and (e)). The increasing sequence $(FM)^k(x)$ converges to $\overline{MF}(x)$. The complements of the sets in the sequence converge to $X' - \overline{MF}(x)$. The following theorem is similar to theorem 6.4.8

Theorem 6.4.14

If $x \in X'$,

$$X' - \overline{MF}(x) \text{ is a c.u.p. set} \tag{a}$$

$$y \in \overline{MF}(x) \Rightarrow \overline{MF}(y) \subset \overline{MF}(x) \tag{b}$$

Proof

a. For every $k \in N$, by theorem 6.4.13(d) and (e),

$X' - (MF)^k(x)$ is a c.u.p. set.

$$X' - \overline{MF}(x) = X' - \bigcup_{k \in N} (MF)^k(x) = \bigcap_{k \in N} (X' - (MF)^k(x))$$

and by theorem 4.3.4, the intersection of c.u.p. sets is a c.u.p. set.

b. follows directly from 6.4.13(c).

We have shown that $\overline{GM}(x)$ and $\overline{MF}(x)$ consist of the points that are revealed preferred to x, respectively to which x is revealed preferred. Between the *interior* points of both sets and x a *strict* (indirect) revealed preference relation holds, which is implied by the strong axiom of revealed preference.

Theorem 6.4.15

If $x \in X'$

$$y \in \operatorname{Int} \overline{GM}(x) \Rightarrow y\bar{\mathsf{P}}x \tag{a}$$

$$y \in \operatorname{Int} \overline{MF}(x) \Rightarrow x\bar{\mathsf{P}}y \tag{b}$$

Proof

a. Let $y \in \operatorname{Int} \underline{G}M(x)$, hence by theorem 6.4.5 (a), $y\bar{\mathsf{R}}x$.
Suppose also $x\bar{\mathsf{R}}y$.
Since $\overline{GM}(x)$ is c.u.p., $\lambda < 1$ exists, such that $\lambda y \in \overline{GM}(x)$ (by theorem 4.3.10), hence $\lambda y\bar{\mathsf{R}}x$. If $y \in G(p)$ and $\lambda y \in G(q)$, $y \in M(p)$ and $y \notin M(q)$, hence $y\mathsf{R}\lambda y$ and $\lambda y \not\mathsf{R} y$, so that $y\mathsf{P}\lambda y$. We have simultaneously $y\mathsf{P}\lambda y$ and $\lambda y\bar{\mathsf{R}}x$, (from $\lambda y\bar{\mathsf{R}}x$ and $x\bar{\mathsf{R}}y$) and by $D7$ this is impossible, hence $y\bar{\mathsf{P}}x$.

b. The argument is similar to that of (a).

Both the sets $\overline{GM}(x)$ and $\overline{MF}(x)$ contain the point x. They may have other points in common, but not interior points.

Theorem 6.4.16

If $x \in X'$

$$\operatorname{Int} \overline{GM}(x) \cap \operatorname{Int} \overline{MF}(x) = \varnothing.$$

Proof

If y would be an element of this intersection, we would have simultaneously $y\bar{\mathsf{P}}x$ and $x\bar{\mathsf{P}}y$, which is a contradiction.

The construction of the sequences $(GM)^k(x)$ and $(MF)^k(x)$ is illustrated in fig. 6.4.17.

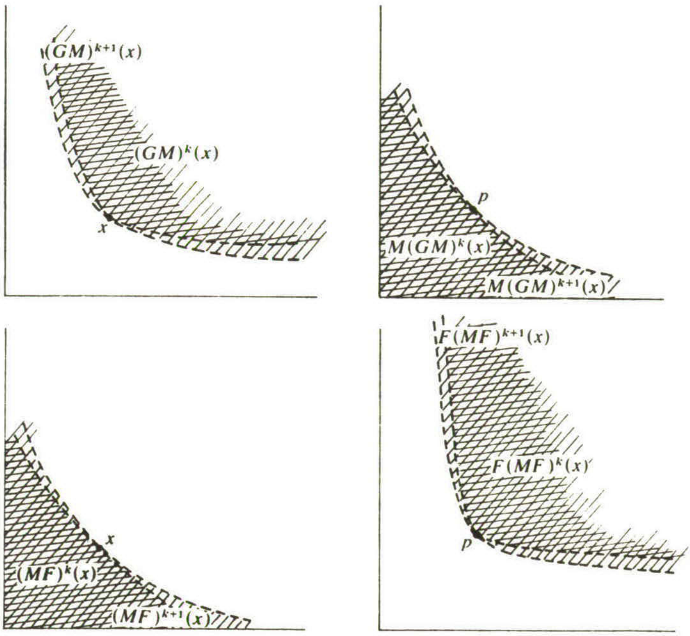


fig. 6.4.17

This development leads to fig. 6.4.18.

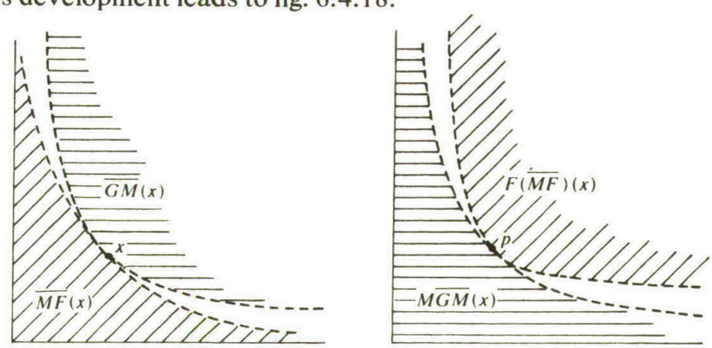


fig. 6.4.18

We can also define correspondences of P' into X' and into itself. From $M: P' \to X'$ and $F: X' \to P'$, we can form composite correspondences such that for every $p \in P'$, we have the sets

$$M(p), FM(p), MFM(p), FMFM(p), \ldots.$$

Similarly from $G: P' \to X'$ and $M: X' \to P'$, we get

$$G(p), MG(p), GMG(p), MGMG(p), \ldots.$$

The set $FM(p)$ e.g. consists of prices at which bundles are demanded, which can also be bought at p. For $q \in FM(p)$, $p\mathrm{R}^*q$ (since for some $y \in M(p)$, we have $q \in F(y)$, hence $y \in G(q)$ and $M(p) \cap G(q) \neq \varnothing$).

The following definitions parallel definitions 6.4.3 and 6.4.10.

Definition 6.4.19

If $p \in P'$ and $k \in N$

$$(FM)^{k+1}(p) = FM(FM)^k(p) \qquad \text{(a)}$$
$$(MG)^{k+1}(p) = MG(MG)^k(p) \qquad \text{(b)}$$

Thus with each $p \in P'$ are associated the following sequences of sets:

in P': $p, FM(p), (FM)^2p, \ldots.$
in X': $M(p), MFM(p), M(FM)^2(p), \ldots.$

and

in P': $p, MG(p), (MG)^2(p), \ldots.$
in X': $G(p), GMG(p), G(MG)^2(p), \ldots.$

The sets $(FM)^k(p)$ and $(MG)^k(p)$ contain prices to which p is revealed at least as favourable in k steps, and that are revealed at least as favourable as p in k steps respectively.

Both sequences are increasing and converge.

Definition 6.4.20

For $p \in P'$

$$\overline{FM}(p) = \bigcup_{k \in N} (FM)^k(p) \qquad \text{(a)}$$

$$\overline{MG}(p) = \bigcup_{k \in N} (MG)^k(p) \qquad \text{(b)}$$

Between the points of these sets and p, the indirect revealed favourability relation holds.

Theorem 6.4.21

If $p \in P'$

$$\overline{FM}(p) = \{q \in P' \mid p\bar{\mathsf{R}}^{\star}q\} \tag{a}$$
$$\overline{MG}(p) = \{q \in P' \mid q\bar{\mathsf{R}}^{\star}p\}. \tag{b}$$

Proof

The argument parallels that of theorems 6.4.5 and 6.4.12.

Theorem 6.4.22

If $p \in P'$

$$\overline{FM}(p) \text{ is a c.u.p. set} \tag{a}$$
$$P' - \overline{MG}(p) \text{ is a c.u.p. set} \tag{b}$$
$$q \in \overline{FM}(p) \Rightarrow \overline{FM}(q) \subset \overline{FM}(p) \tag{c}$$
$$q \in \overline{MG}(p) \Rightarrow \overline{MG}(q) \subset \overline{MG}(p) \tag{d}$$

Proof

See theorems 6.4.8 and 6.4.14 (theorems similar to 6.4.6 and 6.4.13 serve as intermediary results).

Theorem 6.4.23

If $p \in P'$

$$q \in \operatorname{Int} \overline{FM}(p) \Rightarrow p\bar{\mathsf{P}}^{\star}q \tag{a}$$
$$q \in \operatorname{Int} \overline{MG}(p) \Rightarrow q\bar{\mathsf{P}}^{\star}p \tag{b}$$

Theorem 6.4.24

If $p \in P'$

$$\operatorname{Int} \overline{FM}(p) \cap \operatorname{Int} \overline{MG}(p) = \varnothing$$

Thus far we only considered half of the sequences introduced above. Each set of one of the other sequences however, lies between two subsequent sets of the sequences that have been treated (with respect to $\subset$).

Theorem 6.4.25

If $p \in P', x \in G(p)$ and $k \in N$

$$(MG)^k(p) \subset M(GM)^k(x) \subset (MG)^{k+1}(p) \qquad \text{(a)}$$
$$(FM)^k(p) \subset F(MF)^k(x) \subset (FM)^{k+1}(p) \qquad \text{(b)}$$
$$(MF)^k(x) \subset M(FM)^k(p) \subset (MF)^{k+1}(x) \qquad \text{(c)}$$
$$(GM)^k(x) \subset G(MG)^k(p) \subset (GM)^{k+1}(x) \qquad \text{(d)}$$

Proof (only for (a))

$p \in M(x)$, hence by definition 6.4.3

$(MG)^k(p) \subset (MG)^k(M(x)) = M(GM)^k(x)$

and since $x \in G(p)$

$M(GM)^k(x) \subset M(GM)^k(G(p)) = (MG)^{k+1}(p)$

The preceding theorem implies

Theorem 6.4.26

If $P \in P'$ and $x \in G(p)$

$$\overline{MG}(p) = M\overline{GM}(x) \qquad \text{(a)}$$
$$\overline{FM}(p) = F\overline{MF}(x) \qquad \text{(b)}$$
$$\overline{MF}(x) = M\overline{FM}(p) \qquad \text{(c)}$$
$$\overline{GM}(x) = G\overline{MG}(p) \qquad \text{(d)}$$

Proof

$$\overline{MG}(p) = \bigcup_{k \in N} (MG)^k(p) \subset \bigcup_{k \in N} M(GM)^k(x)$$
$$= M\Big(\bigcup_{k \in N} (GM)^k(x)\Big) = M\overline{GM}(x) \subset \bigcup_{k \in N} (MG)^{k+1}(p)$$
$$= \overline{MG}(p)$$

It has been shown that $\overline{GM}(x)$ is a c.u.p. set. In general, it is neither open nor closed. Obviously, the closures of $\overline{GM}(x)$ and $\overline{FM}(p)$ are closed c.u.p. sets (by theorem 4.3.10), whereas the complements of the closures of $\overline{MF}(x)$ and $\overline{MG}(p)$ are open c.u.p. sets.

Definition 6.4.27

If $x \in X'$ and $p \in P'$

$$\overline{\overline{GM}}(x) = \mathrm{Cl}\,\overline{GM}(x) \tag{a}$$

$$\overline{\overline{MF}}(x) = \mathrm{Cl}\,\overline{MF}(x) \tag{b}$$

$$\overline{\overline{FM}}(p) = \mathrm{Cl}\,\overline{FM}(p) \tag{c}$$

$$\overline{\overline{MG}}(p) = \mathrm{Cl}\,\overline{MG}(p) \tag{d}$$

Theorem 6.4.28

If $x \in X'$ and $p \in P'$

$\overline{\overline{GM}}(x)$ and $\overline{\overline{FM}}(p)$ are closed c.u.p. sets (a)

$X' - \overline{MF}(x)$ and $P' - \overline{MG}(p)$ are open c.u.p. sets (b)

Proof

a. By theorem 4.3.10, the closure of a c.u.p. set is a closed c.u.p. set.
b. $X' - \overline{MF}(x)$ is a c.u.p. set (theorem 6.4.14), hence also

$\mathrm{Int}\,(X' - \overline{MF}(x))$ is c.u.p., whereas

$$\mathrm{Int}\,(X' - \overline{MF}(x)) = X' - \mathrm{Cl}\,\overline{MF}(x) = X' - \overline{\overline{MF}}(x)$$

The interiors of the closed sets are identical to the interiors of the original sets.

Theorem 6.4.29

If $x \in X'$ and $p \in P'$

$$\mathrm{Int}\,\overline{GM}(x) = \mathrm{Int}\,\overline{\overline{GM}}(x) \tag{a}$$

$$\mathrm{Int}\,\overline{MF}(x) = \mathrm{Int}\,\overline{\overline{MF}}(x) \tag{b}$$

$$\mathrm{Int}\,\overline{FM}(p) = \mathrm{Int}\,\overline{\overline{FM}}(p) \tag{c}$$

$$\mathrm{Int}\,\overline{MG}(p) = \mathrm{Int}\,\overline{\overline{MG}}(p) \tag{d}$$

Proof

a. $\overline{GM}(x) \subset \mathrm{Cl}\,\overline{GM}(x) = \overline{\overline{GM}}(x)$; hence

$$\mathrm{Int}\,\overline{GM}(x) \subset \mathrm{Int}\,\overline{\overline{GM}}(x)$$

Conversely, let $y \in \operatorname{Int} \overline{\overline{GM}}(x)$ and $y \in G(p)$.

$z \in \operatorname{Int} \overline{\overline{GM}}(x) \cap \operatorname{Int} M(p)$

exists and $\epsilon > 0$ can be found such that

$B_\epsilon(z) \subset \operatorname{Int} \overline{\overline{GM}}(x) \cap \operatorname{Int} M(p)$

and since every neighbourhood contains a point of $\overline{GM}(x)$, a point t exists such that

$t \in B_\epsilon(z) \cap \overline{GM}(x) \cap M(p)$

Since $p \in M(t)$, we have $y \in GM(t) \subset \overline{GM}(x)$, hence

$\operatorname{Int} \overline{\overline{GM}}(x) \subset \operatorname{Int} \overline{GM}(x)$

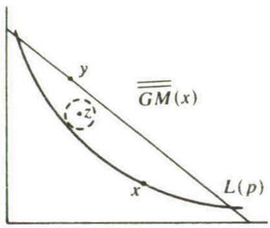


fig. 6.4.30

b. $\operatorname{Int} \overline{MF}(x) \subset \operatorname{Int} \overline{\overline{MF}}(x)$.

If $y \in \operatorname{Int} \overline{\overline{MF}}(x)$, a point $\lambda y \in \operatorname{Int} \overline{\overline{MF}}(x)$ exists, with $\lambda > 1$. By Lemma 4.3.8, ϵ exists such that

$t \in B_\epsilon(\lambda y) \Rightarrow t \geqq y$

Since $B_\epsilon(\lambda y) \cap \overline{MF}(x) \neq \varnothing$, we have $y \in \overline{MF}(x)$, which implies

$\operatorname{Int} \overline{\overline{MF}}(x) \subset \operatorname{Int} \overline{MF}(x)$

This entails that the interiors of $\overline{\overline{GM}}(x)$ and $\overline{\overline{MF}}(x)$ are disjoint (by theorem 6.4.16) and that between the points of these sets and the point x, the *strict* revealed preference relation $\bar{\mathsf{P}}$ holds. Clearly, a similar argument holds for the sets in P'.
The closed sets do not only contain points for which the revealed preference (favourability) relation $\bar{\mathsf{R}}$ ($\bar{\mathsf{R}}^*$) hold with respect to $x(p)$, but also points for which the extended relations $\bar{\bar{\mathsf{R}}}$ and $\bar{\bar{\mathsf{R}}}^*$ hold (see definition 5.9.17). For two points x and y we have $x\bar{\bar{\mathsf{R}}}y$, if either $x \in \overline{\overline{GM}}(y)$ or $y \in \overline{\overline{MF}}(x)$, which mean respectively that every neighbourhood of x

contains a point indirectly revealed preferred to y and that every neighbourhood of y contains a point to which x is indirectly revealed preferred.

Theorem 6.4.31

$$x\bar{\bar{\mathsf{R}}}y \Leftrightarrow x \in \overline{\overline{GM}}(y) \vee y \in \overline{\overline{MF}}(x) \tag{a}$$

$$p\bar{\bar{\mathsf{R}}}^*q \Leftrightarrow p \in \overline{\overline{MG}}(q) \vee q \in \overline{\overline{FM}}(p) \tag{b}$$

Proof

a. By definition 5.9.17, $x\bar{\bar{\mathsf{R}}}y$ if we have either

$\forall\epsilon, \exists z: z \in B_\epsilon(x) \wedge z\bar{\mathsf{R}}y$

or

$\forall\epsilon, \exists z: z \in B_\epsilon(x) \wedge x\bar{\mathsf{R}}z$

The first statement is equivalent to:

$\forall\epsilon, \exists z: z \in B_\epsilon(x) \wedge z \in \overline{GM}(y)$

and by theorem 4.2.14, this holds if and only if

$x \in \mathrm{Cl}\,\overline{GM}(y) = \overline{\overline{GM}}(y)$

The second statement is equivalent to

$\forall\epsilon, \exists z: z \in B_\epsilon(y) \wedge z \in \overline{MF}(x)$

and this is true if and only if

$y \in \mathrm{Cl}\,\overline{MF}(x) = \overline{\overline{MF}}(x)$

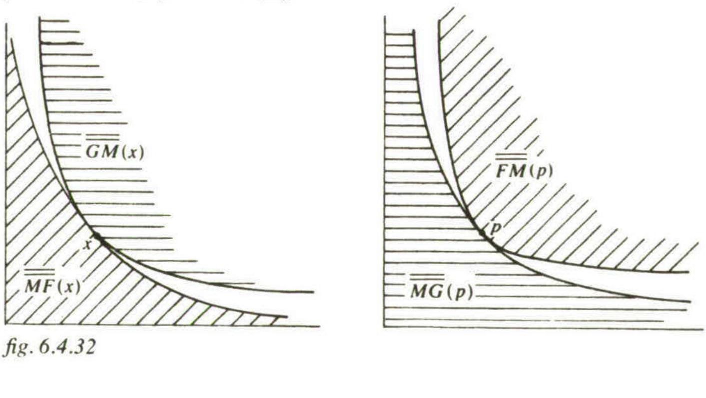


fig. 6.4.32

With each $x \in X'$ correspond two sets in X' and two sets in P', one being c.u.p. and the other having a complement that is a c.u.p. set, whereas their interiors do not intersect. It is shown now, that the c.u.p. set in one space is the *dual* of the complement of the interior of the set in the other space.

Theorem 6.4.33

If $p \in P'$ and $x \in G(p)$,

$$P' - \operatorname{Int} \overline{\overline{MG}}(p) = \overline{\overline{GM}}^*(x) \tag{a}$$

$$X' - \operatorname{Int} \overline{\overline{MF}}(x) = \overline{\overline{FM}}^*(p) \tag{b}$$

Proof

$$\begin{aligned}
\overline{\overline{GM}}^*(x) &= P' - \operatorname{Int} M\overline{\overline{GM}}(x) && \text{(theorem 4.4.30)} \\
\operatorname{Int} M\overline{\overline{GM}}(x) &= M(\operatorname{Int} \overline{\overline{GM}}(x)) && \text{(theorem 4.4.29)} \\
&= M(\operatorname{Int} \overline{GM}(x)) && \text{(theorem 6.4.29)} \\
&= \operatorname{Int} M(\overline{GM}(x)) && \text{(theorem 4.4.29)} \\
&= \operatorname{Int} \overline{MG}(p) && \text{(theorem 6.4.26)} \\
&= \operatorname{Int} \overline{\overline{MG}}(p) && \text{(theorem 6.4.29)}
\end{aligned}$$

The space X' can be partitioned into three sets in two different ways for each x:

$$X' = \operatorname{Int} \overline{\overline{GM}}(x) \cup \operatorname{Bnd} \overline{\overline{GM}}(x) \cup X' - \overline{\overline{GM}}(x)$$

and

$$X' = \operatorname{Int} \overline{\overline{MF}}(x) \cup \operatorname{Bnd} \overline{\overline{MF}}(x) \cup X' - \overline{\overline{MF}}(x)$$

The same is true for P':

$$P' = \operatorname{Int} \overline{\overline{MG}}(p) \cup \operatorname{Bnd} \overline{\overline{MG}}(p) \cup P' - \overline{\overline{MG}}(p)$$

and

$$P' = \operatorname{Int} \overline{\overline{FM}}(p) \cup \operatorname{Bnd} \overline{\overline{FM}}(p) \cup P' - \overline{\overline{FM}}(p)$$

Now by the demand function G, each set of P' corresponds with a set of $\in X'$ for $x \in G(p)$.

Theorem 6.4.35

If $p \in P'$ and $x \in G(p)$,

$$\operatorname{Int} \overline{\overline{GM}}(x) = G(\operatorname{Int} \overline{\overline{MG}}(p)) = G(P' - \overline{\overline{GM}}^{*}(x)) \tag{a}$$

$$\operatorname{Bnd} \overline{\overline{GM}}(x) = G(\operatorname{Bnd} \overline{\overline{MG}}(p)) = G(\operatorname{Bnd} \overline{\overline{GM}}^{*}(x)) \tag{b}$$

$$X' - \overline{\overline{GM}}(x) = G(P' - \overline{\overline{MG}}(p)) = G(\operatorname{Int} \overline{\overline{GM}}^{*}(x)) \tag{c}$$

and

$$\operatorname{Int} \overline{\overline{MF}}(x) = P' - \overline{\overline{FM}}^{*}(p) = G(\operatorname{Int} \overline{\overline{FM}}(p)) \tag{d}$$

$$\operatorname{Bnd} \overline{\overline{MF}}(x) = \operatorname{Bnd} \overline{\overline{FM}}^{*}(p) = G(\operatorname{Bnd} \overline{\overline{FM}}(p)) \tag{e}$$

$$X' - \overline{\overline{MF}}(x) = \operatorname{Int} \overline{\overline{FM}}^{*}(p) = G(P' - \overline{\overline{FM}}(p)) \tag{f}$$

Proof

We only prove (a), (b) and (c).
First the statements (1), (2), (3) and (4) are proved.

1. $G(P' - \overline{\overline{MG}}(p)) \subset X' - \overline{\overline{GM}}(x)$

Let $q \in \operatorname{Int} \overline{\overline{GM}}^{*}(x)$, hence $M(q) \cap \overline{\overline{GM}}^{**}(x) = \emptyset$.

Since $G(q) \subset M(q)$ and $\overline{\overline{GM}}^{**}(x) = \overline{\overline{GM}}(x)$ we have

$G(q) \subset X' - \overline{\overline{GM}}(x)$.

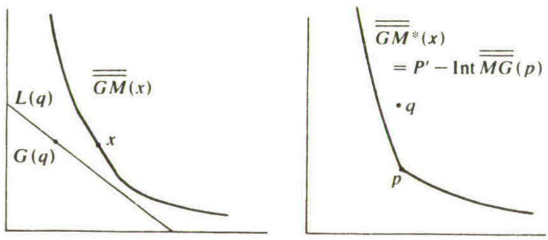


fig. 6.4.36

2. $G(\operatorname{Int} \overline{\overline{MG}}(p)) \subset \overline{\overline{GM}}(x)$

$\operatorname{Int} \overline{\overline{MG}}(p) = \operatorname{Int} \overline{MG}(p)$ (theorem 6.4.29), hence

$G(\operatorname{Int} \overline{\overline{MG}}(p) = G(\operatorname{Int} \overline{MG}(p)) \subset G\overline{MG}(p) \subset \overline{GM}(x) \subset \overline{\overline{GM}}(x)$.

3. $G(\operatorname{Bnd} \overline{\overline{MG}}(p) \subset \operatorname{Bnd} \overline{\overline{GM}}(x)$

Let $q \in \operatorname{Bnd} \overline{\overline{MG}}(p)$. Suppose $y \in G(q)$ and $y \in \overline{\overline{GM}}(x)$. Hence

$\epsilon > 0$ exists, such that $B_\epsilon(y) \cap \overline{\overline{GM}}(x) = \varnothing$. Since $q \in$ Bnd $\overline{\overline{MG}}(p)$, we have for $\lambda < 1$, $\lambda q \in$ Int $\overline{\overline{MG}}(p)$, hence by (2) $G(\lambda q) \subset \overline{\overline{GM}}(x)$. By axiom $D9$, a point $z \in G(\lambda p)$ exists such that

$$\left|z - \frac{1}{\lambda} y\right| < \varphi \left|\frac{1-\lambda}{\lambda}\right| \frac{1}{|q|}$$

hence

$$|y-z| = \left|\frac{1}{\lambda} y - z + y - \frac{1}{\lambda} y\right| < \left|\frac{1}{\lambda} y - z\right| + \left|\frac{1-\lambda}{\lambda}\right| |y| <$$

$$< \varphi \left|\frac{1-\lambda}{\lambda}\right| \frac{1}{|q|} + \left|\frac{1-\lambda}{\lambda}\right| |y| = \left|\frac{1-\lambda}{\lambda}\right| \left(\frac{\varphi}{|q|} + |y|\right)$$

If λ is chosen such that

$$\left|\frac{1-\lambda}{\lambda}\right| \left(\frac{\varphi}{|q|} + |y|\right) < \epsilon$$

we have

$$|z-y| < \epsilon$$

and this is a contradiction, since

$z \in \overline{\overline{GM}}(x)$ and $B_\epsilon(y) \cap \overline{\overline{GM}}(x) = \varnothing$

Hence we have $G(q) \subset \overline{\overline{GM}}(x)$, and since q is a boundary point, it follows

$M(q) \cap \text{Int}\, \overline{\overline{GM}}(x) = \varnothing$

Hence $G(q) \subset \text{Bnd}\, \overline{\overline{GM}}(x)$.

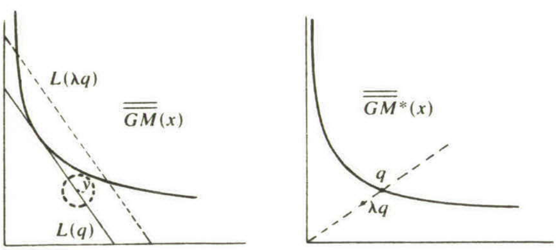


fig. 6.4.37

4. $G(\text{Int}\, \overline{\overline{MG}}(p)) \subset \text{Int}\, \overline{\overline{GM}}(x)$

Let $q \in \text{Int}\, \overline{\overline{MG}}(p)$, by (2), $G(q) \subset \overline{\overline{GM}}(x)$.

Suppose $y \in G(q)$ and $y \in \text{Bnd}\,\overline{\overline{GM}}(x)$.
Since q is an interior point, ϵ exists such that

$$B_\epsilon(q) \cap \overline{\overline{GM}}^*(x) = \varnothing$$

If $\lambda < 1$, we have $\lambda y \in X' - \overline{\overline{GM}}(x)$.
For $\lambda y \in G(r)$, holds $r \in \overline{\overline{GM}}(x) = P' - \text{Int}\,\overline{\overline{MG}}(p)$, since by (2), $r \in \text{Int}\,\overline{\overline{MG}}(p)$ would imply $\lambda y \in G(r) \cap \overline{\overline{GM}}(x)$.
By axiom $D10$, we now have

$$\left|r - \frac{1}{\lambda}q\right| < \Psi\left|\frac{1-\lambda}{\lambda}\right|\frac{1}{|y|}$$

and in the same way as in (3), a contradiction results.

$$\begin{aligned} \text{Now } G(P') &= G(\text{Int}\,\overline{\overline{MG}}(p)) \cup G(\text{Bnd}\,\overline{\overline{MG}}(p)) \\ &\qquad \cup G(P' - \overline{\overline{MG}}(p)) \\ &= X' \\ &= \text{Int}\,\overline{\overline{GM}}(x) \cup \text{Bnd}\,\overline{\overline{GM}}(x) \cup X' - \overline{\overline{GM}}(x) \end{aligned}$$

This implies that the inclusions (1), (3) and (4) must be identities.

This theorem directly implies that the sets in P' correspond to the ones in X' by F, e.g.:

$$F(\text{Int}\,\overline{\overline{GM}}(x)) = \text{Int}\,\overline{\overline{MG}}(p) \quad \text{for } x \in G(p) \tag{6.4.38}$$

It also follows that if $L(q)$ supports $\overline{\overline{GM}}(x)$ in y, then $y \in G(q)$.

Theorem 6.4.39

If $L(q)$ is a supporting hyperplane of $\overline{\overline{GM}}(x)$,

$$L(q) \cap \overline{\overline{GM}}(x) = G(q)$$

Proof

By theorem 6.4.35, a point $y \in \overline{\overline{GM}}(x) \cap L(q)$ exists such that $y \in G(q)$.

Suppose $z \in \overline{\overline{GM}}(x) \cap L(q)$ and $z \notin G(q)$.
By $D4$, $G(q)$ is closed, and therefore a point

$$t = \lambda z + (1-\lambda)y \text{ for } 0 < \lambda < 1$$

exists such that $t \notin G(q)$.

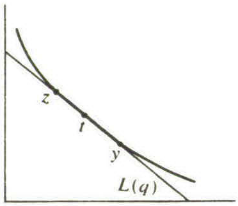


fig. 6.4.39'

Let $t \in G(r)$. Now $L(r) \cap \operatorname{Int} \overline{\overline{GM}}(x) = \varnothing$, hence $z \in L(r)$ and $y \in L(r)$. But then we have

$y \in G(p), t \in M(p), t \notin G(p)$ whereas $y \in M(r)$ and $t \in G(r)$

which contradicts axiom $D6$.

Theorem 6.4.40

If $x \in X'$

$$GM\overline{\overline{GM}}(x) = \overline{\overline{GM}}(x) \qquad \text{(a)}$$

$$MF\overline{\overline{MF}}(x) = \overline{\overline{MF}}(x) \qquad \text{(b)}$$

Proof (for (a))

1. $GM\overline{\overline{GM}}(x) \supset \overline{\overline{GM}}(x)$ (see theorem 6.4.6(b)).

2. Let $x \in G(p)$. Since $\overline{\overline{MG}}(p)$ is the complement of the interior of the dual of $\overline{\overline{GM}}(x)$, for $y \in \overline{\overline{GM}}(x)$, we have $M(y) \subset \overline{\overline{MG}}(p)$, hence $M\overline{\overline{GM}}(x) \subset \overline{\overline{MG}}(p)$.

This implies:

$$GM\overline{\overline{GM}}(x) \subset G\overline{\overline{MG}}(p) = \overline{\overline{GM}}(x)$$

This directly implies

Theorem 6.4.41

$$y \in \overline{\overline{GM}}(x) \Rightarrow \overline{\overline{GM}}(y) \subset \overline{\overline{GM}}(x) \qquad \text{(a)}$$

$$y \in \overline{\overline{MF}}(x) \Rightarrow \overline{\overline{MF}}(y) \subset \overline{\overline{MF}}(x) \qquad \text{(b)}$$

Proof

a. $y \in \overline{\overline{GM}}(x)$, hence $GM(y) \subset GM\overline{\overline{GM}}(x) = \overline{\overline{GM}}(x)$ and hence $\overline{GM}(y) \subset \overline{\overline{GM}}(x)$ and this implies

$\overline{\overline{GM}}(y) = \mathrm{Cl}\,\overline{GM}(y) \subset \mathrm{Cl}\,\overline{\overline{GM}}(x) = \overline{\overline{GM}}(x)$

6.5 THE AXIOMS OF THE CONSUMER PREFERENCE MODEL AS THEOREMS IN THE DEMAND FUNCTION MODEL

In this final section we prove that the axioms of model C except $C1$ and $C2$ are implied by the present model. The exclusion of $D1$ and $D2$ only means that the theory can only be applied to P' and X', the points for which the demand function is defined. Obviously, this is not a severe restriction.

The extended revealed preference relation is complete, which requires that for any $x, y \in X'$, at least one of the relations $y\bar{\bar{R}}x$ or $x\bar{\bar{R}}y$ holds. This is true, if and only if the point y is in at least one of the sets $\overline{\overline{GM}}(x)$ and $\overline{\overline{MF}}(x)$ (see fig. 6.5.1; see also fig. 6.4.32).

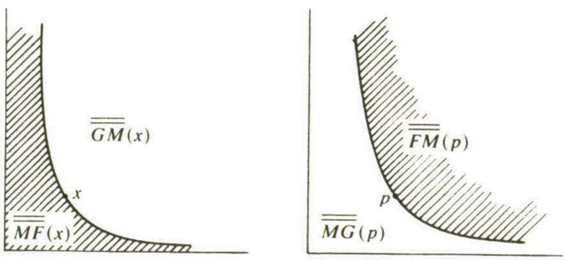


fig. 6.5.1

Hence we must prove

$$X' = \overline{\overline{GM}}(x) \cup \overline{\overline{MF}}(x)$$

or equivalently

$$\overline{\overline{GM}}(x) = X' - \mathrm{Int}\,\overline{\overline{MF}}(x)$$

and the duals of these sets must also coincide

$$\overline{\overline{FM}}(p) = P' - \mathrm{Int}\,\overline{\overline{MG}}(p)$$

or

$$P' = \overline{\overline{FM}}(p) \cup \overline{\overline{MG}}(p)$$

Theorem 6.5.2

$$\forall x \in X' : X' = \overline{\overline{GM}}(x) \cup \overline{\overline{MF}}(x)$$

Proof

Suppose that the theorem is not true and that $x_0 \in X'$ exists, such that

$$[X' - \overline{\overline{MF}}(x_0) - \overline{\overline{GM}}(x_0)] \neq \varnothing$$

Because of theorem 6.4.16, we have

$$\overline{\overline{GM}}(x_0) \subset X' - \overline{\overline{MF}}(x_0)$$

Let $x_0 \in G(p_0)$, then

$$F(\overline{\overline{GM}}(x_0)) = \overline{\overline{MG}}(p_0)$$

and

$$F\overline{\overline{MF}}(x_0) = \overline{\overline{FM}}(p_0)$$

where

$$P' - [\overline{\overline{FM}}(p_0) - \overline{\overline{MG}}(p_0)] \neq \varnothing$$

This set is open and therefore contains a point $r_0 > 0$, and hence also a point $\lambda r_0 = q_0 \in \operatorname{Bnd} \overline{\overline{MG}}(p_0)$, so that for $y_0 \in G(q_0)$ holds $y_0 \in \operatorname{Bnd} \overline{\overline{GM}}(x_0)$ (theorem 6.4.39).

To simplify the notation we define

$$C_1 = \overline{\overline{GM}}(x_0),\ C_1^* = P' - \operatorname{Int} \overline{\overline{MG}}(p_0)$$

$$C_2 = X' - \operatorname{Int} \overline{\overline{MF}}(x_0),\ C_2^* = \overline{\overline{FM}}(p_0)$$

We now have

$C_1 \subset C_2$ (a)

$x_0 \in \operatorname{Bnd} C_1 \cap \operatorname{Bnd} C_2$ and $L(p_0)$ supports C_2 in x_0 (b)

$y_0 \in \operatorname{Bnd} C_1$ and $y_0 \notin \operatorname{Bnd} C_2$ and
$L(q_0)$ supports C_1 in y_0 $(q_0 > 0)$ (c)

These are the conditions of theorem 4.5.10.
They also imply

$C_2^* \subset C_1^*$ (1)

$p_0 \in \operatorname{Bnd} C_1^* \cap \operatorname{Bnd} C_2^*$ (2)

$q_0 \in \operatorname{Bnd} C_1^*$ and $q_0 \notin \operatorname{Bnd} C_2^*$ with $q_0 > 0$ (3)

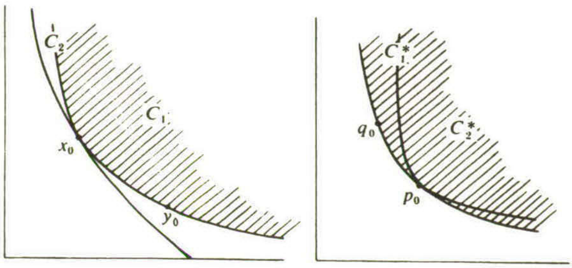


fig. 6.5.3

The theorem will be proved by deriving a contradiction with axiom *D*9. It is required that q lies within a certain distance from p. Therefore, we first prove that the existence of p_0 and q_0 implies the existence of two other points that satisfy the conditions of theorem 4.5.10 and that are sufficiently close together:

Lemma 6.5.4

If p_0 and q_0 are such that (2) and (3) above are satisfied, then p_1 and q_1 exist which also satisfy these conditions, while $\epsilon > 0$ exists such that for some number $\varphi > 0$:

$$q_1 \in B_\epsilon(p_1) \tag{4}$$

$$r \in B_\epsilon(p_1) \Rightarrow |p_1 - r| < \frac{|r|}{\varphi} \tag{5}$$

Proof of the lemma

The set

$V = \{v \in P' \mid v \leqq p_0 + q_0\}$

is closed and bounded and hence also

$W = V \cap \text{Bnd}\, C_1^* \cap \text{Bnd}\, C_2^*$

is closed and bounded. The distance $d(q_0, v)$ attains a minimum in V:

$d(q_0, p_1) = \min_{v \in W} d(q_0, v) > 0,$

so that p_1 is a common boundary point of C_1^* and C_2^*, which is closest to q_0.

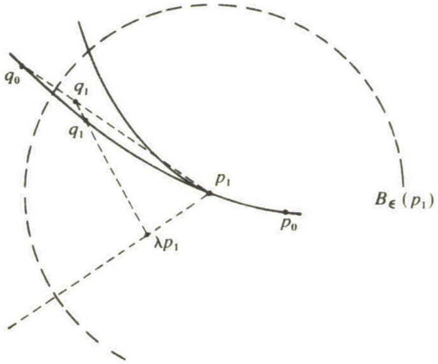


fig. 6.5.5

If $r \in B_\epsilon(p_1)$, we have $|p_1 - r| < \epsilon$, hence $|p_1| < |r| + \epsilon$.

Choose $\epsilon = \dfrac{|p_1|}{\varphi+1}$. This implies

$(\varphi+1)\epsilon = |p_1| < |r| + \epsilon$, hence $\epsilon < \dfrac{|r|}{\varphi}$

and we have

$r \in B_\epsilon(p_1) \Rightarrow |p_1 - r| < \dfrac{|r|}{\varphi}$

Now choose $q_1' = \alpha p_1 + (1-\alpha) q_0$ (for $\alpha \in [0, 1]$), such that

$|q_1' - p_1| = (1-\alpha)|p_1 - q_0| < \epsilon$ with $q_1' \in C_1^*$

If $\lambda < 1$ is chosen such that,

$|p_1 - \lambda p_1| = (1-\lambda)|p_1| < \epsilon$

we have $\lambda p \notin C_1^*$.

The line segment connecting q_1' and λp_1 cuts Bnd C_1^* in a point q_1. Since $B_\epsilon(p_1)$ is convex, $q_1 \in B_\epsilon(p_1)$ (which is condition (4) of the Lemma).

Since

$$\begin{aligned} q_1 &= \gamma_1 q_1' + \gamma_2 \lambda p_1 \\ &= \gamma_1 \alpha p_1 + \gamma_1 (1-\alpha) q_0 + \gamma_2 \lambda p_1 = \delta_1 p_1 + \delta_2 q_0 \end{aligned}$$

(with $\gamma_1 + \gamma_2 = 1, \delta_1 + \delta_2 \leqslant 1$ and $\gamma_1, \gamma_2, \delta_1, \delta_2 \in [0, 1]$)

then taking into account theorem 4.3.11,

$|q_1 - p_1| < |q_0 - p_1|$, hence $q_1 \notin C_2^*$ (condition (3))

Now we are able to finish the proof of theorem 6.5.2:

By theorem 4.5.10, $t_1 \in \text{Bnd}\, C_1$ and $r_1 \in \text{Bnd}\, C_1^*$ exist such that $L(r_1)$ supports C_1 in t_1, hence by theorem 6.4.39, $t_1 \in G(r_1)$ and we have, for $\tau r_1 \in \text{Bnd}\, C_2^*$,

$$t_2 \in G(\tau r_1) \Rightarrow p_1 t_2 \geqslant p_1 t_1$$

However

$$\min_{\mu} |p_1 - \mu r_1| < |p_1 - r_1| < \frac{|r_1|}{\varphi}$$

and this contradicts axiom $D9$ (See theorem 6.3.6).

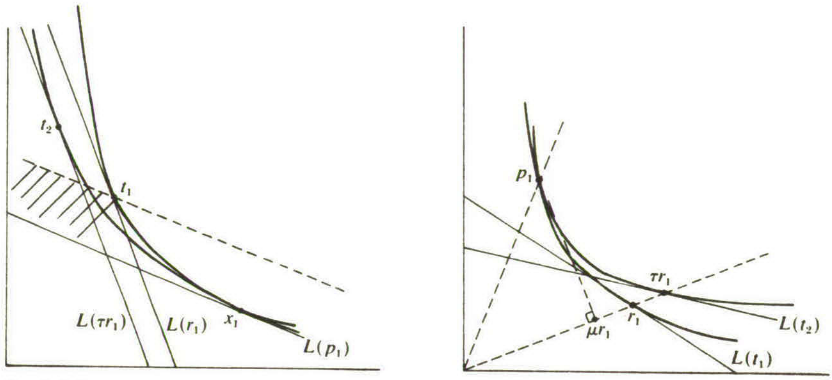


fig. 6.5.6

It follows

Theorem 6.5.7

If $x, y \in X'$,

$$y \in \overline{\overline{GM}}(x) \Leftrightarrow x \in \overline{\overline{MF}}(y)$$

Proof

Let $y \in \overline{\overline{GM}}(x)$. By theorem 6.4.41, $\overline{\overline{GM}}(y) \subset \overline{\overline{GM}}(x)$, hence

$$\text{Int}\, \overline{\overline{GM}}(y) \subset \text{Int}\, \overline{\overline{GM}}(x)$$

and this implies by theorem 6.5.2,

$$\overline{\overline{MF}}(y) \supset \overline{\overline{MF}}(x)$$

and since $x \in \overline{\overline{MF}}(x)$, we have $x \in \overline{\overline{MF}}(y)$.

The converse is proved by a similar argument.

Now we are able to prove that *the extended revealed preference relation* $\bar{\bar{\mathrm{R}}}$ *is a complete preordering.*

Theorem 6.5.8

$$\forall x, y \in X' : x\bar{\bar{\mathrm{R}}}y \vee y\bar{\bar{\mathrm{R}}}x \qquad \text{(a)}$$

$$x\bar{\bar{\mathrm{R}}}y \wedge y\bar{\bar{\mathrm{R}}}z \Rightarrow x\bar{\bar{\mathrm{R}}}z \qquad \text{(b)}$$

Proof

a. By theorem 6.4.2,

$$x \in \overline{\overline{GM}}(y) \vee x \in \overline{\overline{MF}}(y)$$

and by theorem 6.4.31, this is equivalent to

$$x\bar{\bar{\mathrm{R}}}y \vee y\bar{\bar{\mathrm{R}}}x$$

b. By theorems 6.4.31, and 6.5.7,

$$x\bar{\bar{\mathrm{R}}}y \Rightarrow x \in \overline{\overline{GM}}(y)$$

$$y\bar{\bar{\mathrm{R}}}z \Rightarrow y \in \overline{\overline{GM}}(z)$$

By theorem 6.4.41, we have

$$\overline{\overline{GM}}(y) \subset \overline{\overline{GM}}(z)$$

hence

$x \in \overline{\overline{GM}}(z)$, which implies $x\bar{\bar{\mathrm{R}}}z$.

A direct consequence of this theorem is, that points on the boundary of $\overline{\overline{GM}}(x)$ are revealed equivalent.

$$y, z \in \mathrm{Bnd}\,\overline{\overline{GM}}(x) \Rightarrow y\bar{\bar{\mathrm{I}}}z \wedge y\bar{\bar{\mathrm{I}}}x \wedge z\bar{\bar{\mathrm{I}}}x \qquad (6.5.9)$$

It remains to prove that no contradiction can arise between the extended revealed preference relation $\bar{\bar{\mathrm{R}}}$ and the indirect revealed preference relation $\bar{\mathrm{R}}$. Suppose, that for two points x and y would hold $x\bar{\bar{\mathrm{R}}}y$ and $y\bar{\bar{\mathrm{R}}}x$, hence $y\bar{\bar{\mathrm{I}}}x$, and $x\bar{\mathrm{R}}y$ and $y\not{\bar{\mathrm{R}}}x$, hence $x\bar{\mathrm{P}}y$, then $\bar{\bar{\mathrm{R}}}$ would not be a consistent extension of $\bar{\mathrm{R}}$.

By the following theorem, this case is excluded. Note, that this theorem is similar to the strong axiom of revealed preference, which also excluded contradiction between two revealed preference relations.

Theorem 6.5.10

If $x, y \in X'$.

$$x\overset{=}{\mathrm{R}}y \Rightarrow y\not{\bar{\mathrm{P}}}x$$

Proof

Suppose $x\overset{=}{\mathrm{R}}y$ and $y\bar{\mathrm{P}}x$.

Since $x\overset{=}{\mathrm{R}}y$, we have $x \in \overline{\overline{GM}}(y)$. Since $y\bar{\mathrm{P}}x$, we have $y\bar{\mathrm{R}}x$ and hence $y\overset{=}{\mathrm{R}}x$, which implies $x \in \overline{\overline{MF}}(y)$. Therefore $x \in \mathrm{Bnd}\,\overline{\overline{GM}}(y) = \mathrm{Bnd}\,\overline{\overline{MF}}(y)$. Now $y\bar{\mathrm{P}}x$ implies the existence of a chain of elements t_i, such that

$y\mathrm{R}t_1,\ t_1\mathrm{R}t_2, \ldots, t_n\mathrm{R}x$

and by transitivity of $\overset{=}{\mathrm{R}}$

$\forall i\colon t_i\overset{=}{\mathrm{I}}y$

hence

$\forall i\colon t_i \in \mathrm{Bnd}\,\overline{\overline{GM}}(y)$

Let $y \in G(p) = M(p) \cap \overline{\overline{GM}}(y)$ (see theorem 6.4.39).

Since $y\mathrm{R}t_1$, we have $t_1 \in M(p) \cap \overline{\overline{GM}}(y) = G(p)$.
Hence also

$t_1\mathrm{R}y$, which implies $t_1\mathrm{I}y$

By the same argument it follows $t_1\mathrm{I}t_2,\ t_2\mathrm{I}t_3, \ldots, t_n\mathrm{I}x$ hence $x\bar{\mathrm{I}}y$ and this implies $y\not{\bar{\mathrm{P}}}x$, which is a contradiction.

Now the axioms of the consumer preference model are easily proved as theorems of the present model.

Theorem 6.5.11

The preference relation $\gtrsim$ is a complete preordering on X'.

Proof

We show that the relation $\gtrsim$ coincides with $\bar{\bar{\mathsf{R}}}$.
By axiom $D11$(a).

$x\bar{\bar{\mathsf{R}}}y \Rightarrow x \gtrsim y$

If $x\not{\bar{\bar{\mathsf{R}}}}y$ (and, because of completeness, $y\bar{\bar{\mathsf{R}}}x$), then by theorem 6.5.10, $y\bar{\mathsf{P}}x$ must hold which implies by axiom $D11$(b)

$y \succ x$

and hence by definition, $x \not\gtrsim y$. Consequently

$x \gtrsim y \Leftrightarrow x\bar{\bar{\mathsf{R}}}y$

and therefore $\gtrsim$ must also be a complete preordering.

Theorem 6.5.12 (Monotonicity)

If $x, y \in X'$:

$$[x \geqq y \Rightarrow x \gtrsim y] \wedge [x > y \Rightarrow x \succ y]$$

Proof

$x \geqq y \Rightarrow x \in \overline{\overline{GM}}(y) \Rightarrow x\bar{\bar{\mathsf{R}}}y \Rightarrow x \gtrsim y$

$x > y \Rightarrow x \in \operatorname{Int} \overline{\overline{GM}}(y) \Rightarrow x\bar{\mathsf{P}}y \Rightarrow x \succ y$

Theorem 6.5.13 (Convexity)

$\forall x_0 \in X' \colon \{x \in X' \mid x \gtrsim x_0\}$ is convex

Proof

$\{x \mid x \gtrsim x_0\} = \{x \mid x\bar{\bar{\mathsf{R}}}x_0\} = \overline{\overline{GM}}(x)$ is convex.

Theorem 6.5.14 (Continuity (see theorem 5.3.7))

For every $x_0 \in X'$: $\{x \in X' \mid x \gtrsim x_0\}$ and $\{x \in X' \mid x_0 \gtrsim x\}$ are closed

Proof

$\{x \mid x \gtrsim x_0\} = \{x \mid x\bar{\bar{\mathsf{R}}}x_0\} = \overline{\overline{GM}}(x_0)$ is closed

$\{x \mid x_0 \gtrsim x\} = \{x \mid x_0\bar{\bar{\mathsf{R}}}x\} = \overline{\overline{MF}}(x_0)$ is closed

Theorem 6.5.15 (Transition axiom)

$\forall p \in P': H(M(p)) = K(M(p) = G(p)$

Proof

Let $x \in G(p)$.

$G(p) = K(M(p)) = M(p) \cap \text{Bnd}\,\overline{\overline{GM}}(x)$,

where $M(p) \cap \text{Int}\,\overline{\overline{GM}}(x) = \varnothing$.

Hence $M(p) \subset \overline{\overline{MF}}(x)$, and therefore

$y \in M(p) \Rightarrow x\overset{=}{\mathsf{R}}y \Rightarrow x \gtrsim y$

whereas for $y \in M(p)$ and $y \gtrsim x$ holds

$y \in M(p) \cap \text{Bnd}\,\overline{\overline{GM}}(x) = G(p)$

Theorem 6.5.16 (Weak satiation axiom)

$x \sim x+t \wedge t \geqq 0 \Rightarrow [\forall\epsilon, \exists\lambda: z \in B_\epsilon(x) \Rightarrow z+\lambda t \sim z+(\lambda+1)t]$

Proof

This is directly implied by axiom $D8$.

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